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A stochastic PDE system for chemotaxis

Existence of Martingale Solutions

Bachelor's Thesis

submitted by

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Abstract

This thesis studies a stochastic version of the Keller–Segel chemotaxis-consumption system

$$\begin{cases} du = (\Delta u - \operatorname{div}(u \nabla v)) \, dt + u \circ d\mathcal{W}, & \text{in } \mathcal{O} \times (0, T], \\ \partial_t v = \Delta v - uv, & \text{in } \mathcal{O} \times (0, T], \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\mathcal{O} \times (0, T], \\ (u(0), v(0)) = (u_0, v_0), & \text{in } \mathcal{O}, \end{cases} \quad (0.1)$$

on a bounded interval $\mathcal{O} \subset \mathbb{R}$ and for some $T > 0$, with homogeneous Neumann boundary conditions and Stratonovich multiplicative noise, driven by a cylindrical Wiener process $\mathcal{W}$, acting on the cell density u . While the deterministic version is well studied, the stochastic counterpart had, prior to this thesis, not been treated in the literature. We prove the existence of a martingale (probabilistically weak) solution by adapting the truncation and fixed-point construction of Hausenblas, Mukherjee and Tran [HMT22a], developed there for the stochastic production model, to the consumption setting, and we construct a Lyapunov functional suited to the absorption nonlinearity $-uv$ in place of the production nonlinearity these authors work with. Along the way we collect the analytical and stochastic-analytic tools needed for the construction, and we recall the relevant deterministic theory. The structural role of dimensionality is also discussed throughout the thesis.

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1 Introduction

1.1 Motivation

Chemotaxis is the directed movement of cells or organisms along a gradient of a chemical signal. Examples include the aggregation of the slime mold *Dictyostelium discoideum* into a multicellular body under starvation, the guidance of immune cells to sites of infection, and the swimming of aerobic bacteria such as *Bacillus subtilis* towards regions of higher oxygen concentration. The mathematical modeling of chemotaxis at the population level goes back to Patlak [Pat53], who derived a diffusion-drift equation for the density of a population of individually biased random walkers, and to Keller and Segel [KS70; KS71], who coupled this equation to a second one for the evolution of the chemical signal. The resulting *Keller–Segel system* and its variants are among the most studied classes of reaction–diffusion systems in mathematical biology.

Whether solutions exist globally or blow up in finite time depends on the coupling structure, the spatial dimension, and the size of the initial data. All classical results, however, are formulated for deterministic initial data and deterministic dynamics. Real cell populations are subject to randomness: intrinsic fluctuations from the finite number of cells and signal molecules, extrinsic environmental noise (temperature, fluid motion, heterogeneous medium properties), and uncertainty in the parameters governing individual cell behavior. *Stochastic* counterparts of the Keller–Segel system, in which one or several of these sources of randomness are modeled as a stochastic perturbation of the underlying PDE system, have accordingly been studied over the last decade. This thesis contributes to that line of research: we establish an existence theory for a stochastically perturbed version of the chemotaxis-*consumption* model, described in detail below.

1.2 The Two Variants of the Keller–Segel System

Two structurally different variants of the Keller–Segel system are commonly studied, distinguished by the coupling term in the equation for the chemical signal v .

In the classical *production* model, the cell density u produces the chemical signal v , and the system takes the form

$$\begin{cases} \partial_t u = \Delta u - \operatorname{div}(u \nabla v), \\ \partial_t v = \Delta v - v + u, \end{cases}$$

with homogeneous Neumann boundary conditions and initial data. This is the original setting considered by Keller and Segel [KS70] and models, for instance, the aggregation of slime mold amoebae, which secrete a chemoattractant (cyclic AMP) that then diffuses and decays.

In the *consumption* model, which is the object of this thesis, the roles are reversed: the cells consume the chemical signal rather than producing it, leading to a system of the form

$$\begin{cases} \partial_t u = \Delta u - \operatorname{div}(u \nabla v), \\ \partial_t v = \Delta v - uv, \end{cases}$$

again with homogeneous Neumann boundary conditions and initial data. This variant was introduced by Tuval et al. [Tuv+05] to model the chemotactic motion of the aerobic bacterium *Bacillus subtilis* towards higher oxygen concentrations near the air–water interface of a sessile drop, where v represents the dissolved oxygen concentration consumed by the bacteria at a rate proportional to uv . See Lankeit and Winkler’s survey [LW23] for a systematic account of both variants and the differences in their qualitative theory.

1.3 The Role of Dimensionality

For the production model, dimension is decisive. In two space dimensions, solutions exist globally if and only if the total mass $\int_{\mathcal{O}} u_0$ stays below the critical threshold 4π (8π in the radially symmetric case), and blow up in finite time can occur above this threshold [JL92; BDP06]. No such critical-mass dichotomy exists in one dimension, and blow-up may occur for arbitrarily small mass in three and higher dimensions.

The consumption model is more favorable, independently of dimension: since the coupling term $-uv$ is dissipative rather than destabilizing, a maximum principle for v leads to global existence for all $d \leq 3$ [TW12]. This only simplifies matters for $d \geq 2$, however: the strategy adapted here [HMT22a] already works in one dimension, where the production model has no critical-mass obstruction either, so the consumption model isn’t chosen here to sidestep a blow-up difficulty that’s absent in $d = 1$ regardless. Rather, the point is that the techniques developed in this thesis for the consumption nonlinearity do not rely on dimension-specific smallness assumptions, and are hence a starting point for an eventual extension to $d = 2, 3$, where the two models’ behavior genuinely diverges.

1.4 Stochastic Perturbations

To model intrinsic fluctuations of the cell population itself, rather than an external force applied to the system, the natural choice is to add the noise multiplicatively by the state u , interpreting the resulting stochastic integral in the *Stratonovich* sense. This is justified by the Wong–Zakai theorem [WZ65]: Stratonovich stochastic differential equations arise as the limit of ordinary differential equations driven by smooth, physically realizable approximations of white noise.

For the stochastic *production* model, an existence and uniqueness theory has been developed over the last several years: the one-dimensional case [HMT22a; HMT22b], a possible extension to two dimensions [HML22] (work in progress), and porous-medium-type diffusion [MHZ25; HMZ22]. Stochastic chemotaxis–fluid systems, coupling a production- or consumption-type chemotaxis system to a stochastic Navier–Stokes equation, have also been studied [ZZ20;

HJR23; XZL24; ZL23; ZL22]. To the best of my knowledge, no existence theory exists yet for the pure stochastic chemotaxis-*consumption* system (4.1) with noise only in the first component and no fluid coupling. This thesis fills that gap.

1.5 Main Result and Outline

The main result, stated precisely as Theorem 4.6 in Chapter 4, is the existence of a mild martingale solution to the stochastic chemotaxis-consumption system (4.1) on a bounded interval $\mathcal{O} \subset \mathbb{R}$, with homogeneous Neumann boundary conditions and Stratonovich multiplicative noise acting on the cell density u , together with a uniform a priori bound on the solution in expectation. The proof follows the strategy of Hausenblas, Mukherjee and Tran [HMT22a] for the stochastic production model (truncation of the nonlinearity, a Banach-space-valued fixed-point argument for the truncated system, compactness and Skorokhod-type arguments to pass to a martingale solution, and removal of the truncation via a sequence of stopping times), but needs adaptation at essentially every step, since the consumption nonlinearity $-uv$ has a different structure and requires a different Lyapunov functional.

Notational conventions. Throughout, $\mathcal{O} \subset \mathbb{R}^d$ denotes a bounded domain with Lipschitz (in Chapter 2, occasionally convex or C^2) boundary $\partial\mathcal{O}$, and ν its outer unit normal. Generic constants are denoted C and may change from line to line, with dependence on relevant parameters indicated where relevant, e.g. $C = C(T, \mathcal{O})$. Function spaces are abbreviated as $L_{\mathcal{O}}^p := L^p(\mathcal{O})$, $H_{\mathcal{O}}^k := H^k(\mathcal{O})$, and $L_T^p H_{\mathcal{O}}^k := L^p(0, T; H^k(\mathcal{O}))$; the space $L\log L(\mathcal{O})$, written $L\log L_{\mathcal{O}}$, denotes the Zygmund space of functions u with $\int_{\mathcal{O}} |u| \log(1 + |u|) < \infty$. Cylindrical Wiener processes and stochastic integration are used in the sense of [DZ14], as recalled in Sections 2.2–2.4. The symbol $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ denotes a filtration satisfying the usual conditions, and $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a corresponding filtered probability space.

2 Analytical and Stochastic Preliminaries

2.1 Analytical Tools

In this section we will state some important analytical results, which we will use throughout this thesis. Most of the proofs are elementary and can be found in lecture notes, but are nonetheless written out here for completeness and ease of reference.

Lemma 2.1 (Jensen's inequality). *Let $n \in \mathbb{N}$, $f: \mathbb{R} \rightarrow \mathbb{R}$ be convex, $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, and $a_1, \dots, a_n \in \mathbb{R}$. Then*

$$f\left(\sum_{i=1}^n \lambda_i a_i\right) \leq \sum_{i=1}^n \lambda_i f(a_i).$$

Especially the following special case of Lemma 2.1 will be used often in inequalities with norms.

Corollary 2.2. *For $n \in \mathbb{N}$, $1 \leq p \leq \infty$ (mostly used for $n \in \{2, 3\}$ and $p = 2$) and $a_1, \dots, a_n \in \mathbb{R}$ we have*

$$\left|\sum_{i=1}^n a_i\right|^p \leq n^{p-1} \sum_{i=1}^n |a_i|^p.$$

Corollary 2.3 (Jensen's inequality for expectations). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X \in L^q(\Omega)$ a random variable, and $f: \mathbb{R} \rightarrow \mathbb{R}$ convex. Then*

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Corollary 2.4 (Jensen's inequality for integrals). *Let $g \in L^q(0, T)$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ convex. Then*

$$f\left(\frac{1}{T} \int_0^T g(s) \, ds\right) \leq \frac{1}{T} \int_0^T f(g(s)) \, ds.$$

Proof of Lemma 2.1. By induction on n . For $n = 2$: since f is convex,

$$f(\lambda_1 a_1 + \lambda_2 a_2) \leq \lambda_1 f(a_1) + \lambda_2 f(a_2)$$

holds by definition of convexity. The general case follows by induction: assuming the inequality holds for $n - 1$ points, write

$$\sum_{i=1}^n \lambda_i a_i = (1 - \lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} a_i + \lambda_n a_n,$$

apply convexity and then the induction hypothesis to $\sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} a_i$. $\square$

Proof of Corollary 2.2. Since $x \mapsto |x|^p$ is convex, Jensen's inequality (Lemma 2.1) yields

$$\left| \sum_{i=1}^n a_i \right|^p = n^p \left| \sum_{i=1}^n \frac{1}{n} a_i \right|^p \leq n^p \sum_{i=1}^n \frac{1}{n} |a_i|^p = n^{p-1} \sum_{i=1}^n |a_i|^p. \quad \square$$

Proof of Corollary 2.3. Apply Lemma 2.1 to the discrete approximation of $\mathbb{E}[X]$ and pass to the limit, using that f is convex and hence continuous; see e.g. [DZ14, Chapter 6]. $\square$

Proof of Corollary 2.4. Apply Corollary 2.3 to the uniform distribution on $[0, T]$, i.e., set $\mathbb{P} = \frac{1}{T} \lambda|_{[0, T]}$ and $X = g$, where λ denotes the Lebesgue measure. $\square$

Lemma 2.5 (Young's inequality). *Let $a, b \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (2.1)$$

In particular, for any $\varepsilon > 0$,

$$|AB| \leq \varepsilon A^p + C_Y(\varepsilon) B^q, \quad C_Y(\varepsilon) = \frac{1}{q} (\varepsilon p)^{-\frac{q}{p}}. \quad (2.2)$$

Proof. For $a = 0$ or $b = 0$ the statement is obviously true. Henceforth we assume $a, b > 0$ and we observe that $-\log(\cdot)$ is convex. Jensen's inequality (Lemma 2.1) yields

$$-\log \left(\frac{a^p}{p} + \frac{b^q}{q} \right) \leq -\frac{1}{p} \log a^p - \frac{1}{q} \log b^q = -\log ab,$$

which after multiplying with -1 and exponentiating is equivalent to (2.1).

For (2.2) substitute $a \mapsto \lambda a$, $b \mapsto \lambda^{-1} b$ with $\lambda = (\varepsilon p)^{1/p}$ into (2.1) to obtain

$$ab \leq \frac{\lambda^p}{p} a^p + \frac{\lambda^{-q}}{q} b^q = \varepsilon a^p + \frac{(\varepsilon p)^{-q/p}}{q} b^q. \quad \square$$

Theorem 2.6 (Hölder inequality). *Let $(\mathcal{O}, \mathcal{F}, \mu)$ be a measure space, f, g be μ -measurable functions and $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}. \quad (2.3)$$

Proof. Obviously we can assume $\|f\|_{L^p}, \|g\|_{L^q} > 0$. For $p = 1$, $q = \infty$ the statement follows directly from $|fg| \leq \|g\|_{L^\infty} |f|$ a.e. For $p, q \in (1, \infty)$ we get by using Young's inequality (2.1) with $a = \frac{|f|}{\|f\|_{L^p}}$ and $b = \frac{|g|}{\|g\|_{L^q}}$ the following:

$$\frac{1}{\|f\|_{L^p} \|g\|_{L^q}} \int_{\mathcal{O}} |fg| = \int_{\mathcal{O}} ab \leq \int_{\mathcal{O}} \left(\frac{a^p}{p} + \frac{b^q}{q} \right) = \frac{1}{p \|f\|_{L^p}^p} \int_{\mathcal{O}} |f|^p + \frac{1}{q \|g\|_{L^q}^q} \int_{\mathcal{O}} |g|^q = \frac{1}{p} + \frac{1}{q} = 1. \quad \square$$

Theorem 2.7 (Minkowski's inequality). *Let $(\mathcal{O}, \mathcal{F}, \mu)$ be a measure space, $p \in [1, \infty]$, and $f, g \in L^p(\mathcal{O})$. Then*

$$\|f + g\|_{L^p(\mathcal{O})} \leq \|f\|_{L^p(\mathcal{O})} + \|g\|_{L^p(\mathcal{O})}. \quad (2.4)$$

Proof. First note that by Corollary 2.2, $f + g$ has finite L^p -norm. If this norm is zero the statement is true, hence assume $\|f + g\|_{L^p} > 0$. For $p = \infty$ the statement is elementary. For $p < \infty$ using the Hölder inequality (Theorem 2.6) with exponents p and $\frac{p}{p-1}$, we obtain

$$\begin{aligned} \|f + g\|_{L^p}^p &= \int_{\mathcal{O}} |f + g|^p = \int_{\mathcal{O}} |f + g| \cdot |f + g|^{p-1} \\ &\leq \int_{\mathcal{O}} |f| \cdot |f + g|^{p-1} + \int_{\mathcal{O}} |g| \cdot |f + g|^{p-1} \\ &\leq \left(\left(\int_{\mathcal{O}} |f|^p \right)^{\frac{1}{p}} + \left(\int_{\mathcal{O}} |g|^p \right)^{\frac{1}{p}} \right) \cdot \left(\int_{\mathcal{O}} |f + g|^{(p-1)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &= (\|f\|_{L^p} + \|g\|_{L^p}) \|f + g\|_{L^p}^{p-1}. \quad \square \end{aligned}$$

Lemma 2.8 (Minkowski's inequality for Bochner integrals). *Let E be a Banach space and $f \in L^1(0, T; E)$. Then*

$$\left\| \int_0^T f(t) dt \right\|_E \leq \int_0^T \|f(t)\|_E dt. \quad (2.5)$$

Proof. With the dual space E^* of E we find that

$$\begin{aligned} \left\| \int_0^T f(t) dt \right\|_E &= \sup_{\|\phi\|_{E^*}=1} \left| \phi \left(\int_0^T f(t) dt \right) \right| = \sup_{\|\phi\|_{E^*}=1} \left| \int_0^T \phi(f(t)) dt \right| \\ &\leq \sup_{\|\phi\|_{E^*}=1} \int_0^T |\phi(f(t))| dt \leq \sup_{\|\phi\|_{E^*}=1} \int_0^T \|\phi\|_{E^*} \|f(t)\|_E dt = \int_0^T \|f(t)\|_E dt. \quad \square \end{aligned}$$

Lemma 2.9 (Markov inequality). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $f: (\Omega, \mathcal{A}) \rightarrow (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ and $r, \varepsilon > 0$. Then it holds that*

$$\mu(\{|f| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^r} \int |f|^r d\mu.$$

Especially for a probability measure $\mu = \mathbb{P}$ and a random variable $f = X$ we get

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{1}{\varepsilon^r} \mathbb{E}[|X|^r].$$

Proof. Let $A(\varepsilon) := \{|f| \geq \varepsilon\} \in \mathcal{A}$. Then

$$\int |f|^r d\mu \geq \int \mathbb{1}_{A(\varepsilon)} |f|^r d\mu \geq \int \mathbb{1}_{A(\varepsilon)} \varepsilon^r d\mu = \varepsilon^r \mu(A(\varepsilon))$$

Theorem 2.10 (Gagliardo–Nirenberg interpolation inequality). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $p, q, r \in [1, \infty]$, $j < m$ and $\theta \in [j/m, 1]$ such that*

$$\frac{1}{p} = \frac{j}{d} + \theta \left(\frac{1}{r} - \frac{m}{d} \right) + \frac{1-\theta}{q}.$$

Then there exists $C > 0$ depending only on $\mathcal{O}$, d , m , j , p , q , r such that

$$\|D^j u\|_{L^p} \leq C \|D^m u\|_{L^r}^\theta \|u\|_{L^q}^{1-\theta} + C \|u\|_{L^q} \quad (2.6)$$

for all $u \in W^{m,r}(\mathcal{O}) \cap L^q(\mathcal{O})$. (Unlike on $\mathbb{R}^d$, where the term $C\|u\|_{L^q}$ is not needed, on a bounded domain it is required in general, since there is no scaling argument available.)

Proof. This is the bounded-domain version of the classical Gagliardo–Nirenberg inequality; see [Eva10, Section 5.8, Theorem 5] for a full proof (extension to a ball, the inequality on $\mathbb{R}^d$ via a scaling/covering argument, and the extra zero-order term arising from the extension operator), or [BS88] for the general interpolation-theoretic viewpoint. $\square$

The following cases of the Gagliardo–Nirenberg inequality will be used often.

Corollary 2.11. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $d \in \{1, 2, 3, 4\}$,*

$$\|w\|_{L^4} \leq C \|\nabla w\|_{L^2}^{\frac{d}{4}} \|w\|_{L^2}^{1-\frac{d}{4}} + C \|w\|_{L^2} \quad \text{for } w \in H^1(\mathcal{O}), \quad (2.7)$$

$$\|\nabla w\|_{L^4} \leq C \|D^2 w\|_{L^2}^{\frac{1}{2}(\frac{d}{4}+1)} \|w\|_{L^2}^{\frac{1}{2}(1-\frac{d}{4})} + C \|w\|_{L^2} \quad \text{for } w \in H^2(\mathcal{O}). \quad (2.8)$$

Proof. For $p = 4$, $q, r = 2$, $j = 0$, $m = 1$ we get

$$\frac{1}{4} = \theta \left(\frac{1}{2} - \frac{1}{d} \right) + \frac{1-\theta}{2} = \frac{1}{2} - \frac{\theta}{d} \implies \theta = \frac{d}{4} \in [0, 1],$$

and for the same p, q, r but $j = 1$, $m = 2$ we can see that

$$\frac{1}{4} = \frac{1}{d} + \theta \left(\frac{1}{2} - \frac{2}{d} \right) + \frac{1-\theta}{2} = \frac{1}{d} + \frac{1}{2} - \frac{2\theta}{d} \implies \theta = \frac{1}{2} \left(\frac{d}{4} + 1 \right) \in [0, 1]. \quad \square$$

Theorem 2.12 (Poincaré inequality). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $p \in (1, \infty)$. There exists $C = C(\mathcal{O}) > 0$ such that*

$$\|w - \bar{w}\|_{L^p(\mathcal{O})} \leq C \|\nabla w\|_{L^p(\mathcal{O})} \quad \text{for all } w \in W^{1,p}(\mathcal{O}), \quad (2.9)$$

where $\bar{w} = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} w$ denotes the mean value of w .

Proof. Suppose, the statement is false. Then for every $n \in \mathbb{N}$ there would exist $w_n \in W^{1,p}(\mathcal{O})$ with $\|w_n - \bar{w}_n\|_{L^p} > n \|\nabla w_n\|_{L^p}$ and WLOG $\|w_n - \bar{w}_n\|_{L^p} = 1$ and $\bar{w}_n = 0$ (one may consider $\tilde{w}_n = \frac{w_n - \bar{w}_n}{\|w_n - \bar{w}_n\|_{L^p}}$). We conclude $\|\nabla w_n\|_{L^p} \leq \frac{1}{n}$ and for a subsequence

$$w_{n_k} \rightharpoonup w \text{ in } W^{1,p}(\mathcal{O}), \quad w_{n_k} \rightarrow w \text{ in } L^p(\mathcal{O}) \quad \text{and} \quad \|\nabla w_{n_k}\|_{L^p} \rightarrow 0,$$

such that $w \in W^{1,p}(\mathcal{O})$ with $\nabla w \equiv 0$. Hence for a standard mollifier ϕ_ε we have $\nabla(\phi_\varepsilon \star w) =$

$\phi_\varepsilon \star \nabla w = 0$ which with Fubini leads to

$$\begin{aligned} \int_{\mathcal{O}} (\phi_\varepsilon \star w)(x) \, dx &= \int_{\mathcal{O}} \int_{\mathbb{R}^d} \phi_\varepsilon(x-y) w(y) \, dy \, dx = \int_{\mathbb{R}^d} \int_{\mathcal{O}} \phi_\varepsilon(x-y) w(y) \, dx \, dy \\ &= \int_{\mathbb{R}^d} w(y) \int_{\mathcal{O}} \phi_\varepsilon(x-y) \, dx \, dy = \int_{\mathcal{O}} w(y) \int_{\mathcal{O}} \phi_\varepsilon(x-y) \, dx \, dy \\ &\xrightarrow{\varepsilon \rightarrow 0} \int_{\mathcal{O}} w(y) \cdot 1 \, dy = \bar{w} \cdot |\mathcal{O}|, \end{aligned}$$

so $\phi_\varepsilon \star w \rightarrow c$ for $\varepsilon \rightarrow 0$ for some constant $c \in \mathbb{R}$. Since $\phi_\varepsilon \star w \rightarrow w$ in $L^p(\mathcal{O})$ as $\varepsilon \rightarrow 0$, we conclude $w \equiv c$ a.e. in $\mathcal{O}$. Since $w_{n_k} \rightarrow w$ in $L^p(\mathcal{O})$, the mean values satisfy $\bar{w}_{n_k} \rightarrow \bar{w} = c$. But $\bar{w}_{n_k} = 0$ for all k , so $c = 0$, i.e. $w \equiv 0$. However, since $w_{n_k} \rightarrow w = 0$ in $L^p(\mathcal{O})$, we have

$$1 = \|w_{n_k} - \bar{w}_{n_k}\|_{L^p} = \|w_{n_k}\|_{L^p} \rightarrow 0,$$

a contradiction. Hence the Poincaré inequality holds. $\square$

Lemma 2.13 (Sobolev embeddings). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $k \in \mathbb{N}$, $1 \leq p < \infty$. Then we have the following embeddings*

(i) *Subcritical case ($kp < d$):*

$$W^{k,p}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O}) \quad \text{for all } 1 \leq q \leq p^* := \frac{dp}{d-kp}.$$

(ii) *Critical case ($kp = d$):*

$$W^{k,p}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O}) \quad \text{for all } 1 \leq q < \infty.$$

(iii) *Supercritical case ($kp > d$):*

$$W^{k,p}(\mathcal{O}) \hookrightarrow C^{0,\alpha}(\bar{\mathcal{O}}) \quad \text{for } \alpha = k - \frac{d}{p} \in (0, 1),$$

and in particular

$$W^{k,p}(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O}).$$

(iv) *Hilbert space case ($p = 2$, $s > d/2$):*

$$H^s(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O}) \quad \text{for } s > \frac{d}{2}.$$

In particular, for $d = 1$: $H^1(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$, and for $d \in \{2, 3\}$: $H^2(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$.

(v) *Compact embeddings (Rellich–Kondrachov): If $kp \leq d$,*

$$W^{k,p}(\mathcal{O}) \xhookrightarrow{C} L^q(\mathcal{O}) \quad \text{for all } 1 \leq q < p^*,$$

and if $kp > d$,

$$W^{k,p}(\mathcal{O}) \xhookrightarrow{C} C^0(\bar{\mathcal{O}}).$$

(vi) *Negative Sobolev spaces:* For $s > \frac{d}{2}$,

$$L^1(\mathcal{O}) \hookrightarrow H^{-s}(\mathcal{O}).$$

Here, as throughout this thesis, $H^{-s}(\mathcal{O}) := (H^s(\mathcal{O}))'$ denotes the dual of the full Sobolev space $H^s(\mathcal{O})$ (paired via the $L^2(\mathcal{O})$ inner product), not the dual of $H_0^s(\mathcal{O})$. The embedding follows directly from (iv): since $\mathcal{O}$ is a bounded Lipschitz domain, $H^s(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ for $s > d/2$, so $f \in L^1(\mathcal{O})$ defines a bounded functional $\phi \mapsto \int_{\mathcal{O}} f \phi$ on $H^s(\mathcal{O})$ with $|\langle f, \phi \rangle| \leq \|f\|_{L^1} \|\phi\|_{L^\infty} \leq C \|f\|_{L^1} \|\phi\|_{H^s}$.

All embeddings are continuous.

Proof. See [Eva10, Chapter 5] and [AF03, Chapter 6] for (i)–(v); (vi) is the direct duality argument given above. $\square$

Lemma 2.14 (Interpolation in positive Sobolev spaces). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $0 \leq \rho \leq 1$. Then there exists $C = C(\mathcal{O}, \rho) > 0$ such that*

$$\|f\|_{H^\rho} \leq C \|f\|_{L^2}^{1-\rho} \|f\|_{H^1}^\rho$$

for all $f \in H^1(\mathcal{O})$.

Lemma 2.15 (Interpolation in negative Sobolev spaces). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $0 < \delta < \varepsilon$. Then there exists $C > 0$ such that*

$$\|f\|_{H^{-\delta}} \leq C \|f\|_{H^{-\varepsilon}}^{\delta/\varepsilon} \|f\|_{L^2}^{1-\delta/\varepsilon}$$

for all $f \in L^2(\mathcal{O})$, where $H^{-\delta}(\mathcal{O})$, $H^{-\varepsilon}(\mathcal{O})$ are understood in the sense of Lemma 2.13(vi).

Lemma 2.16 (Product estimate in Sobolev spaces). *Let $\mathcal{O} \subset \mathbb{R}$, $\delta \in (0, \frac{1}{2})$ and $\rho = \delta + \frac{1}{2}$. Then there exists $C > 0$ such that*

$$\|fg\|_{L^2(\mathcal{O})} \leq C \|f\|_{H^{-\delta}(\mathcal{O})} \|g\|_{H^\rho(\mathcal{O})}$$

for all $f \in H^{-\delta}(\mathcal{O})$ and $g \in H^\rho(\mathcal{O})$.

Proof. We can calculate

$$\begin{aligned} \|fg\|_{L^2} &= \sup_{\|\phi\|_{L^2}=1} \langle f, g\phi \rangle \\ &\stackrel{(1)}{\leq} \sup_{\|\phi\|_{L^2}=1} \|f\|_{H^{-\delta}} \|g\phi\|_{H^\delta} \\ &\stackrel{(2)}{\leq} \sup_{\|\phi\|_{L^2}=1} \|f\|_{H^{-\delta}} \cdot C \|g\|_{H^\rho} \|\phi\|_{L^2} \\ &= C \|f\|_{H^{-\delta}} \|g\|_{H^\rho}, \end{aligned}$$

where the inequalities hold because of the following arguments:

- (1) Definition of the dual norm $\|f\|_{H^{-\delta}} = \sup_{\|\psi\|_{H^\delta}=1} \langle f, \psi \rangle$, applied with $\psi = g\phi / \|g\phi\|_{H^\delta}$.
- (2) Multiplication estimate: Since $\rho = \delta + \frac{1}{2} > \frac{1}{2}$, we have $H^\rho(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ when $d = 1$, and g hence is a bounded multiplier on $L^2(\mathcal{O})$. The interpolation identity $H^\delta = [L^2, H^1]_\delta$ (complex interpolation, see [Tri95, Chapter 4] or [BS88, Chapter 4]) yields $\|g\phi\|_{H^\delta} \leq C\|g\|_{H^\rho}\|\phi\|_{L^2}$.

See [RS96, p. 238]. □

Theorem 2.17 (Gronwall's inequality, differential form). *Let $y: [0, T] \rightarrow \mathbb{R}$ be absolutely continuous with $y(0) \leq y_0$ and*

$$y'(t) \leq \alpha(t) + \beta(t)y(t) \quad \text{for a.e. } t \in [0, T],$$

where $\beta, \alpha \in L^1(0, T)$. Then

$$y(t) \leq e^{\int_0^t \beta(s) ds} \left(y_0 + \int_0^t \alpha(s) e^{-\int_0^s \beta(r) dr} ds \right) \quad \text{for all } t \in [0, T]. \quad (2.10)$$

Proof. Define the integrating factor

$$\mu(t) := e^{-\int_0^t \beta(s) ds}.$$

Since $\beta \in L^1(0, T)$, μ is absolutely continuous, positive, and satisfies $\mu'(t) = -\beta(t)\mu(t)$ for a.e. $t \in [0, T]$. Set $z(t) := \mu(t)y(t)$. Then z is absolutely continuous and, for a.e. t ,

$$z'(t) = \mu'(t)y(t) + \mu(t)y'(t) = \mu(t)(y'(t) - \beta(t)y(t)) \leq \mu(t)\alpha(t)$$

Since μ is bounded and $\alpha \in L^1(0, T)$, $\mu\alpha$ is integrable. Integrating from 0 to t gives

$$z(t) \leq z(0) + \int_0^t \mu(s)\alpha(s) ds = \mu(0)y(0) + \int_0^t \alpha(s) e^{-\int_0^s \beta(r) dr} ds \leq y_0 + \int_0^t \alpha(s) e^{-\int_0^s \beta(r) dr} ds,$$

using $\mu(0) = 1$ and $y(0) \leq y_0$. Dividing by $\mu(t) > 0$ yields (2.10). □

Let us collect some more versions of the Gronwall inequality in the following.

Lemma/Corollary 2.18. *Let $y_0 \in \mathbb{R}$, $\rho \in (0, 1)$, $C_1 \in \mathbb{R}$, $C_2 \geq 0$ and $\alpha, \beta_1, \beta_2 \in L^1(0, T)$. Then for absolutely continuous $y: [0, T] \rightarrow [0, \infty)$ with $y(0) \leq y_0$ the following statements hold.*

(i) *If $y'(t) \leq C_1 + C_2 y(t)^\rho$ or $y(t) \leq C_1 + C_2 \int_0^t y(s)^\rho ds$ for a.e. $t \in [0, T]$, then*

$$y(t) \leq C(\rho, C_1, C_2, T, y_0) \quad \forall t \in [0, T].$$

(ii) *If $y'(t) \leq \beta_1(t)y(t) + \beta_2(t)y(t)^\rho$ or $y(t) \leq y_0 + \int_0^t \beta_1(s)y(s) ds + \int_0^t \beta_2(s)y(s)^\rho ds$ for a.e. $t \in [0, T]$, then*

$$y(t) \leq e^{\int_0^t \beta_1(s) ds} \left(y_0^{1-\rho} + (1-\rho) \int_0^t \beta_2(s) e^{-(1-\rho)\int_0^s \beta_1(\sigma) d\sigma} ds \right)^{\frac{1}{1-\rho}}.$$

(iii) If $y(t) \leq C_1 + C_2 \int_0^t (t-s)^{\rho-1} y(s) ds$ for a.e. $t \in [0, T]$, then

$$y(t) \leq C_1 C(\rho, T, C_2).$$

(iv) All inequalities from before also hold for y of the form

$$y(t) = \mathbb{E} \left[\sup_{s \in [0, t]} |X(s)|^2 \right],$$

when we use the integral formulation. For example when proven the first inequality here

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X(s)|^2 \right] \leq C_1 + C_2 \mathbb{E} \left[\sup_{s \in [0, t]} \int_0^s |X(\sigma)|^2 d\sigma \right] \leq C_1 + C_2 \int_0^t \mathbb{E} \left[\sup_{\sigma \in [0, t]} |X(\sigma)|^2 \right] ds,$$

we can use (ii) or Theorem 2.17.

Proof. (i) Since $\rho < 1$, the function $g(y) := C_1 + C_2 y^\rho$ is continuous and the ODE $z'(t) = g(z(t))$, $z(0) = y_0$, has a global solution on $[0, T]$ by the comparison principle. By separation of variables,

$$\int_{y_0}^{z(t)} \frac{d\sigma}{C_1 + C_2 \sigma^\rho} = t,$$

and since $\rho < 1$ the left-hand side diverges as $z \rightarrow \infty$, so $z(t) < \infty$ for all $t \in [0, T]$. By the comparison principle $y(t) \leq z(t) \leq C(\rho, C_1, C_2, T, y_0)$.

(ii) Since $\rho < 1$, the map $z(t) := y(t)^{1-\rho}$ is well-defined and, wherever $y(t) > 0$, absolutely continuous with $z'(t) = (1-\rho)y(t)^{-\rho}y'(t) \leq (1-\rho)\beta_1(t)y(t)^{1-\rho} + (1-\rho)\beta_2(t) = (1-\rho)\beta_1(t)z(t) + (1-\rho)\beta_2(t)$ for a.e. t with $y(t) > 0$; on $\{y = 0\}$ we trivially have $z = 0$ and the same differential inequality holds since $z' = 0$ there (a.e., by the a.c. chain rule for $t \mapsto y(t)^{1-\rho}$). Applying Gronwall's inequality (Theorem 2.17) with $\alpha(t) = (1-\rho)\beta_2(t)$, $\beta(t) = (1-\rho)\beta_1(t)$ and $z(0) = y_0^{1-\rho}$ yields

$$z(t) \leq e^{(1-\rho) \int_0^t \beta_1(s) ds} \left(y_0^{1-\rho} + (1-\rho) \int_0^t \beta_2(s) e^{-(1-\rho) \int_0^s \beta_1(\sigma) d\sigma} ds \right),$$

and raising both sides to the power $\frac{1}{1-\rho}$ gives the claim.

(iii) See [Hen81, Theorem 1, p. 188] or [Hen81, Lemma 7.1.1]. □

Lemma 2.19 (Semigroup estimates). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup on $\mathcal{O}$. Let $\lambda_1 > 0$ denote the first positive eigenvalue of $-\Delta$ under Neumann boundary conditions. Then there exists a constant $C > 0$ depending only on $\mathcal{O}$ with the following properties.*

(i) **Contraction:** For $1 \leq p \leq \infty$ and $w \in L^p(\mathcal{O})$,

$$\|e^{t\Delta} w\|_{L^p_{\mathcal{O}}} \leq \|w\|_{L^p_{\mathcal{O}}}. \quad (2.11)$$

(ii) **Spectral contraction of $(e^{t\Delta} - I)$:** For $w \in L^2(\mathcal{O})$,

$$\|(e^{t\Delta} - I)w\|_{L^2_{\mathcal{O}}} \leq (1 - e^{-\lambda_1 t})\|w\|_{L^2_{\mathcal{O}}} \quad \text{for all } t > 0.$$

(iii) **L^p - L^q smoothing:** For $1 \leq q \leq p \leq \infty$ and $w \in L^q(\mathcal{O})$ with $\int_{\mathcal{O}} w = 0$,

$$\|e^{t\Delta}w\|_{L^p_{\mathcal{O}}} \leq C \left(1 + t^{-\frac{d}{2} - \frac{1}{q} + \frac{1}{p}}\right) e^{-\lambda_1 t} \|w\|_{L^q_{\mathcal{O}}} \quad \text{for all } t > 0.$$

(iv) **Gradient smoothing:** For $1 \leq q \leq p \leq \infty$ and $w \in L^q(\mathcal{O})$,

$$\|\nabla e^{t\Delta}w\|_{L^p_{\mathcal{O}}} \leq C \left(1 + t^{-\frac{1}{2} - \frac{d}{2} - \frac{1}{q} + \frac{1}{p}}\right) e^{-\lambda_1 t} \|w\|_{L^q_{\mathcal{O}}} \quad \text{for all } t > 0.$$

(v) **Gradient contraction:** For $2 \leq p < \infty$ and $w \in W^{1,p}(\mathcal{O})$,

$$\|\nabla e^{t\Delta}w\|_{L^p_{\mathcal{O}}} \leq C e^{-\lambda_1 t} \|\nabla w\|_{L^p_{\mathcal{O}}} \quad \text{for all } t > 0.$$

(vi) **Smoothing with divergence:** For $1 < q \leq p < \infty$ and $w \in (C_c^\infty(\mathcal{O}))^d$, (compactly supported, since $C_c^\infty(\mathcal{O})$ is dense in $L^q(\mathcal{O})$ for $1 \leq q < \infty$, which is what is needed below to extend $e^{t\Delta} \operatorname{div}$ to all of $(L^q(\mathcal{O}))^d$ by density)

$$\|e^{t\Delta} \operatorname{div} w\|_{L^p_{\mathcal{O}}} \leq C \left(1 + t^{-\frac{1}{2} - \frac{d}{2} - \frac{1}{q} + \frac{1}{p}}\right) e^{-\lambda_1 t} \|w\|_{(L^q_{\mathcal{O}})^d} \quad \text{for all } t > 0.$$

Consequently, for all $t > 0$ the operator $e^{t\Delta} \operatorname{div}$ possesses a uniquely determined extension to an operator form $(L^q(\mathcal{O}))^d$ to $L^p(\mathcal{O})$.

(vii) **Sobolev smoothing:** For $w \in L^2(\mathcal{O})$, $\alpha \geq 0$, and $t > 0$,

$$\|e^{t\Delta}w\|_{H^\alpha_{\mathcal{O}}} \leq C t^{-\alpha/2} \|w\|_{L^2_{\mathcal{O}}}.$$

(viii) **Hölder regularity in time:** For $w \in H^{2\theta}(\mathcal{O})$, $\theta \in (0, 1)$, and $t > 0$,

$$\|(e^{t\Delta} - I)w\|_{L^2_{\mathcal{O}}} \leq C t^\theta \|w\|_{H^{2\theta}}.$$

(ix) **Fractional powers:** For $w \in L^2(\mathcal{O})$, $\alpha > 0$, and $t > 0$,

$$\|(-\Delta)^\alpha e^{t\Delta}w\|_{L^2_{\mathcal{O}}} \leq C t^{-\alpha} \|w\|_{L^2_{\mathcal{O}}}.$$

(x) **Generalized Sobolev smoothing:**¹ For $w \in H^{-2\rho}(\mathcal{O})$ and $\alpha, \rho \in \mathbb{R}$ with $\alpha + 2\rho > 0$,

$$\|e^{t\Delta}w\|_{H^\alpha_{\mathcal{O}}} \leq C t^{-\frac{\alpha+2\rho}{2}} \|w\|_{H^{-2\rho}} \quad \text{for all } t > 0.$$

¹This estimate, generalizing (iii), (vi) and (vii) to arbitrary real Sobolev indices, was first found with the assistance of the AI system Claude and subsequently verified against the source cited below.

(xi) **Generalized Hölder regularity in time:** For $w \in H^{\alpha+2\theta}(\mathcal{O})$, $\alpha \geq 0$, $\theta \in (0, 1)$,

$$\|(e^{t\Delta} - I)w\|_{H_{\mathcal{O}}^{\alpha}} \leq C t^{\theta} \|w\|_{H_{\mathcal{O}}^{\alpha+2\theta}} \quad \text{for all } t > 0.$$

In particular, the existing estimate (viii) is the special case $\alpha = 0$.

Proof. For (i): Let ϕ be the solution to $\partial_t \phi = \Delta \phi$ in $\mathcal{O} \times (0, T)$, $\partial_{\nu} \phi|_{\partial \mathcal{O}} = 0$ in $(0, T)$, $\phi(\cdot, 0) = w$ in $\mathcal{O}$ (hence $\phi(t) = e^{t\Delta} w$). Then

$$\frac{1}{p} \frac{d}{dt} \|\phi(t)\|_{L^p}^p = \int_{\mathcal{O}} |\phi|^{p-2} \phi \phi_t = \int_{\mathcal{O}} |\phi|^{p-2} \phi \Delta \phi = -(p-1) \int_{\mathcal{O}} |\phi|^{p-2} |\nabla \phi|^2 \leq 0 \quad (2.12)$$

and hence

$$\|e^{t\Delta} w\|_{L^p} = \|\phi(t)\|_{L^p} \leq \|\phi(0)\|_{L^p} = \|w\|_{L^p}. \quad (2.13)$$

See [Win10, Lemma 1.3] for (ii)–(v) and [Paz83, Chapter 2] and [Hen81, Chapter 7] for (vi)–(viii). Estimates (ix) and (x), stated for general real Sobolev indices, follow from the same analytic-semigroup fractional-power calculus, see [Paz83, Chapter 2, in particular Theorem 2.6.13]; (x) is the special case $\alpha = 0$ recovering (viii), as already noted above. $\square$

Lemma 2.20. *Let $\mathcal{O}$ be convex and bounded, $w \in C^2(\bar{\mathcal{O}})$ with $\partial_{\nu} w|_{\partial \mathcal{O}} = 0$. Then*

$$\partial_{\nu} |\nabla w|^2 \leq 0. \quad (2.14)$$

Proof. Let $x_0 \in \partial \mathcal{O}$. We have $\nabla w(x_0) \cdot \nu = 0$, and hence $\nabla w(x_0) \in T_{x_0} \partial \mathcal{O}$. Since $\mathcal{O}$ is convex, there exists $t_0 > 0$ and a path $x \in C^1([0, t_0]; \partial \mathcal{O})$ with $x(0) = x_0$ and $\partial_t x(0) = \nabla w(x_0)$. Since $\nu(x(t)) \cdot \nabla w(x(t)) = 0$ for all $t \in [0, t_0]$ differentiation with respect to t yields

$$(D\nu)(x(t)) \cdot \partial_t x(t) \cdot \nabla w(x(t)) + \nu(x(t)) \cdot (D^2 w)(x(t)) \cdot \partial_t x(t) = 0 \quad \forall t \in (0, t_0) \quad (2.15)$$

and at $t = 0$, substituting $\partial_t x(0) = \nabla w(x_0)$, we get

$$(D\nu)(x_0) \cdot \nabla w(x_0) \cdot \nabla w(x_0) + \nu(x_0) \cdot (D^2 w)(x_0) \cdot \nabla w(x_0) = 0. \quad (2.16)$$

Since $\mathcal{O}$ is convex, the shape operator $D\nu$ of $\partial \mathcal{O}$ is positive semi-definite (see e.g. [Car76]), and $D^2 w$ is symmetric, so we can conclude

$$\frac{1}{2} \partial_{\nu} |\nabla w|^2 \Big|_{x_0} = \nabla w \cdot (D^2 w) \cdot \nu \Big|_{x_0} = -(\nabla w(x_0))^{\top} \cdot (D\nu)(x_0) \cdot \nabla w(x_0) \leq 0$$

Lemma 2.21 ($D^2 \log w$ -identity for Neumann functions). *Let $\mathcal{O} \subset \mathbb{R}^d$ be bounded with C^2 boundary and let $w \in C^2(\bar{\mathcal{O}})$ with $w > 0$ in $\bar{\mathcal{O}}$ and $\partial_{\nu} w|_{\partial \mathcal{O}} = 0$. No convexity of $\mathcal{O}$ is required for this lemma. Then the following two identities hold.*

(i) *(Pointwise algebraic identity.)*

$$\int_{\mathcal{O}} w |D^2 \log w|^2 dx = \int_{\mathcal{O}} \left[\frac{|D^2 w|^2}{w} - \frac{|\nabla w|^4}{w^3} + \frac{|\nabla w|^2 \Delta w}{w^2} \right] dx. \quad (2.17)$$

(ii) (*Boundary identity.*)

$$-\int_{\mathcal{O}} \frac{|\Delta w|^2}{w} dx = \int_{\mathcal{O}} \left[-\frac{|D^2 w|^2}{w} - \frac{3}{2} \frac{|\nabla w|^2 \Delta w}{w^2} + \frac{|\nabla w|^4}{w^3} \right] dx + \frac{1}{2} \int_{\partial \mathcal{O}} \frac{1}{w} \partial_\nu |\nabla w|^2 dx. \quad (2.18)$$

If, in addition, $\mathcal{O}$ is convex, then by Lemma 2.20 the boundary term in (2.18) satisfies $\int_{\partial \mathcal{O}} \frac{1}{w} \partial_\nu |\nabla w|^2 dx \leq 0$.

Proof. (i) A direct computation gives the pointwise matrix identity

$$|D^2 \log w|^2 = \frac{|D^2 w|^2}{w^2} + \frac{|\nabla w|^4}{w^4} - 2 \frac{D^2 w : (\nabla w \otimes \nabla w)}{w^3},$$

where the last term can be integrated by parts (using $\nabla |\nabla w|^2 = 2D^2 w \cdot \nabla w$) after multiplying with w

$$-2 \int_{\mathcal{O}} \frac{(D^2 w \cdot \nabla w) \cdot \nabla w}{w^2} dx = - \int_{\mathcal{O}} \frac{\nabla |\nabla w|^2 \cdot \nabla w}{w^2} dx = \int_{\mathcal{O}} \frac{|\nabla w|^2 \Delta w}{w^2} dx - 2 \int_{\mathcal{O}} \frac{|\nabla w|^4}{w^3} dx,$$

which, multiplied by w and integrated, gives (2.17).

(ii) By [Win12, Lemma 3.1], for every $h \in C^2(\mathbb{R})$ and every $\varphi \in C^2(\bar{\mathcal{O}})$ with $\partial_\nu \varphi|_{\partial \mathcal{O}} = 0$,

$$\begin{aligned} \int_{\mathcal{O}} h'(\varphi) |\nabla \varphi|^2 \Delta \varphi dx &= -\frac{2}{3} \int_{\mathcal{O}} h(\varphi) |\Delta \varphi|^2 dx + \frac{2}{3} \int_{\mathcal{O}} h(\varphi) |D^2 \varphi|^2 dx - \frac{1}{3} \int_{\mathcal{O}} h''(\varphi) |\nabla \varphi|^4 dx \\ &\quad - \frac{1}{3} \int_{\partial \mathcal{O}} h(\varphi) \partial_\nu |\nabla \varphi|^2 dx. \end{aligned}$$

Applying this with $\varphi = w$ and $h(s) = 1/s$ (so $h'(s) = -1/s^2$, $h''(s) = 2/s^3$) and rearranging yields exactly (2.18). $\square$

Definition 2.22. Let X be a topological space, $M \subseteq X$ and $r: X \rightarrow M$ continuous. M is called **retract** of X and r is called **retraction**, if $r|_M = \text{id}_M$, i.e., $r(x) = x$ for all $x \in M$.

Lemma 2.23. For any $x_0 \in \mathbb{R}^d$ for $d \in \mathbb{N}$ and any $R > 0$ $\partial B_R(x_0)$ is not a retract of $\overline{B_R(x_0)} \subseteq \mathbb{R}^d$.

Proof. The proof of this statement is not as simple and straightforward as it seems. It can be found in [Zei86, Proposition 2.5] and is only sketched here.

Without loss of generality we set $x_0 = 0$. Suppose there exists a retraction $r: \overline{B_R} \rightarrow \partial B_R$. Then the functions $f(x) = x - r(x)$ and $g(x) = 0$ have the same values on ∂B_R and hence have the same amount of fixpoints in B_R , which is only one at $x = 0$. Hence $r(0) = 0 \notin \partial B_R$ which is a contradiction. $\square$

Theorem 2.24 (Brouwer fixed point). Let $M \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, nonempty, convex, compact and let $f: M \rightarrow M$ be continuous. Then f has a fixpoint.

Proof. The proof is split into two cases.

- 1) Let $R > 0$. Suppose $f: \overline{B_R} \rightarrow \overline{B_R}$ would have no fixed point. Then we could construct a mapping which maps x to the ray from $f(x)$ through x . Exactly one point in ∂B_R would

lie on this ray, call it $r(x)$. We would hence have a continuous function $r: \overline{B_R} \rightarrow \partial B_R$ with $r(x) = x$ for all $x \in \partial B_R$. This r would be a retraction from $\overline{B_R}$ to ∂B_R which is a contradiction to Lemma 2.23. Hence f has a fixed point.

- 2) For any $M \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, we can find a ball $\overline{B_R}$ such that $M \subseteq \overline{B_R}$ and a retraction $r: \overline{B_R} \rightarrow M$. Then with Case 1 $f \circ r: \overline{B_R} \rightarrow M \subseteq \overline{B_R}$ has a fixed point $x \in \text{im}(f \circ r) \subseteq M$, which by definition of r as a retraction fulfills $r(x) = x$, and hence $x = f(r(x)) = f(x)$.

□

Definition 2.25. A set M is called precompact if $\overline{M}$ is compact. This is equivalent to the property that every sequence $(x_n)_{n \in \mathbb{N}} \subseteq M$ has a subsequence converging in $\overline{M}$.

Definition 2.26. Let X, Y be Banach spaces and $T: \text{dom}(T) \subseteq X \rightarrow Y$ be an operator. T is called compact if T is continuous and maps bounded sets into precompact sets.

Lemma 2.27 (Mazur). *Let X be a Banach space and $K \subset X$ compact. Then $\overline{\text{conv}(K)}$ is compact in X .*

Proof. Since X is complete, it suffices to show that $\overline{\text{conv}(K)}$ is totally bounded. Let $\varepsilon > 0$. Since K is compact, it is totally bounded, so there exist $x_1, \dots, x_n \in K$ with $K \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$. Let $C := \text{conv}\{x_1, \dots, x_n\}$, the continuous image of the compact simplex $\{\lambda \in [0, 1]^n : \sum_i \lambda_i = 1\}$ under $\lambda \mapsto \sum_i \lambda_i x_i$, hence itself compact.

We claim $\text{conv}(K) \subseteq C + \overline{B_\varepsilon(0)}$. Let $x = \sum_{j=1}^m \mu_j y_j \in \text{conv}(K)$ with $y_j \in K$, $\mu_j \geq 0$, $\sum_j \mu_j = 1$. For each j choose i_j with $y_j \in B_\varepsilon(x_{i_j})$, i.e. $y_j = x_{i_j} + z_j$ with $\|z_j\|_X < \varepsilon$. Then

$$x = \sum_{j=1}^m \mu_j x_{i_j} + \sum_{j=1}^m \mu_j z_j,$$

where the first sum lies in C (a convex combination of the x_i 's) and the second has norm $\leq \sum_j \mu_j \|z_j\|_X < \varepsilon$, proving the claim. Since $C + \overline{B_\varepsilon(0)}$ is closed (a compact set plus a closed ball), $\overline{\text{conv}(K)} \subseteq C + \overline{B_\varepsilon(0)}$.

As C is compact, it has a finite ε -net $c_1, \dots, c_k$, so $C + \overline{B_\varepsilon(0)} \subseteq \bigcup_{l=1}^k B_{2\varepsilon}(c_l)$. Hence $\overline{\text{conv}(K)}$ is covered by finitely many balls of radius 2ε . Since $\varepsilon > 0$ was arbitrary, $\overline{\text{conv}(K)}$ is totally bounded, and being closed in the complete space X , it is itself complete. A complete, totally bounded metric space is compact. □

Theorem 2.28 (Schauder fixed point). *Let X be a Banach space, $M \subseteq X$ nonempty, convex, compact and $T: M \rightarrow M$ continuous. Then T has a fixed point.*

Theorem 2.29 (Schauder fixed point). *Let X be a Banach space, $M \subseteq X$ nonempty, bounded, closed, convex and $T: M \rightarrow M$ compact. Then T has a fixed point.*

Proof of Theorem 2.28. Let $n \in \mathbb{N}$. Since $T(M)$ is precompact, we find finitely many $y_1, \dots, y_N \in T(M)$ such that $T(M) \subseteq \bigcup_{i=1}^N B_{\frac{1}{n}}(y_i)$ and define

$$P_n(x) := \sum_{i=1}^N a_i(x) y_i / \sum_{i=1}^N a_i(x), \quad a_i(x) := \left(\frac{1}{n} - \|Tx - y_i\|_X \right)_+.$$

Then $\dim \text{span im}(P_n) \leq N$ and

$$\|P_n x - Tx\|_X = \left\| \frac{\sum_{i=1}^N a_i(x)(y_i - Tx)}{\sum_{i=1}^N a_i(x)y_i} \right\|_X \leq \left\| \frac{\sum_{i=1}^N a_i(x)\frac{1}{n}}{\sum_{i=1}^N a_i(x)y_i} \right\|_X = \frac{1}{n} \quad \forall x \in M.$$

For the convex hull $M_n = \text{conv}(\{y_1, \dots, y_N\})$ (see Lemma 2.27) we have $M_n \subseteq \text{conv}(T(M)) \subseteq M$ since M is convex, $P_n: M_n \rightarrow M_n$ is continuous and $M_n \subseteq \mathbb{R}^N$ convex und compact.

With Brouwer's fixed point theorem (Theorem 2.24) there exists $x_n = P_n(x_n)$ and since M is compact there exists $x \in M$ and a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $x_{n_k} \rightarrow x$ and

$$\|x_n - Tx\|_X \leq \underbrace{\|P_n x_n - Tx_n\|_X}_{\leq \frac{1}{n}} + \underbrace{\|Tx_n - Tx\|_X}_{T \text{ continuous}} \xrightarrow{n \rightarrow \infty} 0,$$

hence $x = Tx$. □

Proof of Theorem 2.29. Let $A := \overline{\text{conv}(T(M))} \subseteq M$. Then A is convex and compact with $T(A) \subseteq A$. Hence $T: A \rightarrow A$ has a fixed point in $A \subseteq M$. □

Lemma 2.30. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $F \in L^2(0, T; L^2(\mathcal{O}))$ and $w_0 \in L^2(\mathcal{O})$. If w fulfills

$$\begin{cases} \partial_t w = \Delta w + F(w), & \text{in } [0, T] \times \mathcal{O}, \\ \partial_\nu w = 0, & \text{on } [0, T] \times \partial\mathcal{O}, \\ w(0, \cdot) = w_0, & \text{on } \mathcal{O}, \end{cases}$$

then the maximal regularity holds, i.e., there exists a constant $C > 0$ such that

$$\|w\|_{L_T^\infty H_\mathcal{O}^1}^2 + \|w\|_{L_T^2 H_\mathcal{O}^2}^2 \leq C \left(\|w_0\|_{L_\mathcal{O}^2}^2 + \|F(w)\|_{L_T^2 L_\mathcal{O}^2}^2 \right).$$

Proof. We can calculate with Young's inequality

$$\frac{1}{2} \partial_t \|w(t)\|_{L^2}^2 \leq \frac{1}{2} \partial_t \|w(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 = \int_{\mathcal{O}} w(t) F(w(t)) \leq \frac{1}{2} \|w(t)\|_{L^2}^2 + \frac{1}{2} \|F(w(t))\|_{L^2}^2,$$

as well as

$$\frac{1}{2} \partial_t \|\nabla w(t)\|_{L^2}^2 + \|\Delta w(t)\|_{L^2}^2 = - \int_{\mathcal{O}} \Delta w(t) F(w(t)) \leq \frac{1}{2} \|\Delta w(t)\|_{L^2}^2 + \frac{1}{2} \|F(w(t))\|_{L^2}^2,$$

which leads to

$$\partial_t \left(\|\nabla w(t)\|_{L^2}^2 + \int_0^t \|\Delta w(s)\|_{L^2}^2 ds \right) \leq \|F(w(t))\|_{L^2}^2.$$

Using Gronwall we obtain

$$\|w(t)\|_{L^2}^2 \leq e^T \left(\|w_0\|_{L^2}^2 + \int_0^t \|F(w(s))\|_{L^2}^2 ds \right),$$

and

$$\|\nabla w(t)\|_{L^2}^2 + \int_0^t \|\Delta w(s)\|_{L^2}^2 ds \leq \|w_0\|_{L^2}^2 + \int_0^t \|F(w(s))\|_{L^2}^2 ds,$$

which proves the claim. $\square$

2.2 Probabilistic Setting and Cylindrical Wiener Processes

Throughout this thesis, $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denotes a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. We always assume that $\mathfrak{A}$ satisfies the *usual conditions*, i.e. that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, that $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets of $\mathcal{F}$, and that $\mathbb{F}$ is right-continuous, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T)$. These conditions guarantee, among other things, that hitting times of closed sets by continuous adapted processes are stopping times.

A stochastic process $X: [0, T] \times \Omega \rightarrow E$ with values in a Banach space E is called $\mathbb{F}$ -*adapted* if $X(t)$ is $\mathcal{F}_t$ -measurable for every $t \in [0, T]$.

A stochastic process $X: [0, T] \times \Omega \rightarrow E$ with values in a Banach space E is called *progressively measurable* with respect to $\mathbb{F}$ if for every $t \in [0, T]$ the restriction $X|_{[0, t] \times \Omega}$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable. Every $\mathbb{F}$ -adapted process with $\mathbb{P}$ -a.s. right-continuous (in particular continuous) paths is progressively measurable.

Before introducing the cylindrical (infinite-dimensional) Wiener process used throughout this thesis, we recall the classical real-valued building blocks it is assembled from.

A real-valued process $\beta = (\beta(t))_{t \in [0, T]}$ on $\mathfrak{A}$ is called a (standard, real) $(\mathcal{F}_t)$ -*Wiener process* (or Brownian motion) if $\beta(0) = 0$ $\mathbb{P}$ -a.s., β has $\mathbb{P}$ -a.s. continuous paths, β is $(\mathcal{F}_t)$ -adapted, and for all $0 \leq s \leq t \leq T$ the increment $\beta(t) - \beta(s)$ is independent of $\mathcal{F}_s$ and normally distributed with mean 0 and variance $t - s$. For a progressively measurable real process X with $\mathbb{E}[\int_0^T X(s)^2 ds] < \infty$, the (real) *Itô integral* $\int_0^t X(s) d\beta(s)$ is defined as the $L^2(\Omega)$ -limit of Riemann–Itô sums $\sum_i X(t_i)(\beta(t_{i+1} \wedge t) - \beta(t_i \wedge t))$ over partitions of $[0, t]$ with mesh tending to zero (using the *left* endpoint $X(t_i)$, in contrast to the Stratonovich integral of Section 2.6). It defines a continuous, square-integrable real martingale with $\mathbb{E}[(\int_0^t X d\beta)^2] = \mathbb{E}[\int_0^t X(s)^2 ds]$; see [DZ14].

Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $(e_k)_{k \in \mathbb{Z}}$. A *cylindrical Wiener process* $\mathcal{W}$ on $\mathcal{H}$ over $\mathfrak{A}$ is formally given by the (not $\mathcal{H}$ -valued, but well-defined as a sum of independent real Wiener processes) series

$$\mathcal{W}(t) = \sum_{k \in \mathbb{Z}} \beta_k(t) e_k, \quad t \in [0, T],$$

where $(\beta_k)_{k \in \mathbb{Z}}$ is a family of independent real-valued standard $(\mathcal{F}_t)$ -Wiener processes.

Given a further separable Hilbert space K (in this thesis, typically $K = L^2(\mathcal{O})$), an operator $\Phi \in L(\mathcal{H}, K)$ is called *Hilbert–Schmidt* if $\|\Phi\|_{\text{HS}}^2 := \sum_{k \in \mathbb{Z}} \|\Phi e_k\|_K^2 < \infty$. The space of such operators, equipped with $\|\cdot\|_{\text{HS}}$, is itself a separable Hilbert space, denoted $L_2(\mathcal{H}, K)$ [DZ14,

Appendix C]. For a progressively measurable process $\Phi: [0, T] \times \Omega \rightarrow L_2(\mathcal{H}, K)$ with $\mathbb{E}[\int_0^T \|\Phi(s)\|_{\mathcal{H}}^2 ds] < \infty$, the stochastic integral $\int_0^t \Phi(s) d\mathcal{W}(s)$ is defined as the K -valued process

$$\int_0^t \Phi(s) d\mathcal{W}(s) := \sum_{k \in \mathbb{Z}} \int_0^t \Phi(s) e_k d\beta_k(s), \quad t \in [0, T],$$

which is a well-defined square-integrable K -valued martingale satisfying the Itô isometry $\mathbb{E}[\|\int_0^t \Phi d\mathcal{W}\|_K^2] = \mathbb{E}[\int_0^t \|\Phi(s)\|_{\mathcal{H}}^2 ds]$, and its maximal moments are controlled by the Burkholder–Davis–Gundy inequality of Theorem 2.35 below.

For $p \geq 1$, a Banach space E , and $\mathfrak{R} \in \{\mathfrak{A}, \mathfrak{A}_n \mid n \in \mathbb{N}\}$ a (fixed or truncation-indexed) filtered probability space of the type above, with sample space $\Omega_{\mathfrak{R}}$ and expectation $\mathbb{E}_{\mathfrak{R}}$ (so $\Omega_{\mathfrak{A}} = \Omega$, $\mathbb{E}_{\mathfrak{A}} = \mathbb{E}$, and $\Omega_{\mathfrak{A}_n} = \Omega_n$, $\mathbb{E}_{\mathfrak{A}_n} = \mathbb{E}_n$), we write

$$\begin{aligned} \mathcal{M}_{\mathfrak{R}}^p(0, T; E) &:= \left\{ X: [0, T] \times \Omega_{\mathfrak{R}} \rightarrow E \text{ progressively measurable w.r.t. the filtration of } \mathfrak{R} \mid \right. \\ &\quad \left. \|X\|_{\mathcal{M}_{\mathfrak{R}}^p(0, T; E)}^p := \mathbb{E}_{\mathfrak{R}} \left[\sup_{t \in [0, T]} \|X(t)\|_E^p \right] < \infty \right\}, \end{aligned}$$

a Banach space under $\|\cdot\|_{\mathcal{M}_{\mathfrak{R}}^p(0, T; E)}$ (two processes being identified if they agree $\mathfrak{R}$ -a.s. for a.e. t). This is precisely the space denoted $\mathcal{M}_{\mathfrak{A}}^2(0, T; L^2(\mathcal{O}))$ and $\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ for its truncation-level- n analogue, used throughout Chapter 4 (e.g. in (4.10) and Theorem 2.35 above) as the natural Banach space of stochastic processes on which the fixed-point argument of Section 2.7 is carried out.

2.3 Solution Concepts for SPDEs

We collect here, following [DZ14], the notions of solution for an abstract stochastic evolution equation

$$dX(t) = (AX(t) + F(X(t))) dt + G(X(t)) d\mathcal{W}(t), \quad X(0) = X_0, \quad (2.19)$$

on a Hilbert space K , where A generates a C_0 -semigroup $(e^{tA})_{t \geq 0}$ on K (in our application, $K = L^2(\mathcal{O})$ and $A = \Delta$ or $A = \Delta + \gamma I$, cf. Lemma 2.19), $F: K \rightarrow K$, $G: K \rightarrow L_2(\mathcal{H}, K)$, and $\mathcal{W}$ is a cylindrical Wiener process as in Section 2.2. This is the abstract framework instantiated by (4.1) after the conversion of Section 4.2.1.

A word on what (2.19) actually *means* is in order, since it is easy to misread by analogy with a deterministic PDE. For a deterministic evolution equation $\dot{u}(t) = Au(t) + F(u(t))$, the notion of a *classical* solution is unambiguous: u is (continuously) differentiable in t , and the equation holds pointwise for every t . This notion does not carry over to (2.19), because $\mathcal{W}$ is, $\mathbb{P}$ -a.s., nowhere differentiable in t (this is a standard property of Brownian motion, inherited by every component β_k of $\mathcal{W}$; see e.g. [DZ14]) – so there is no function “ $d\mathcal{W}/dt$ ” for which (2.19) could hold pointwise in the classical sense, no matter how regular X , A , F , G are. Consequently, (2.19) is, by definition, only ever *notation* for the corresponding integral

equation

$$X(t) = X_0 + \int_0^t (AX(s) + F(X(s))) ds + \int_0^t G(X(s)) dW(s), \quad t \in [0, T],$$

where the second integral is the Itô integral of Section 2.2, and analogously for the weak and mild formulations of Definition 2.32 below. Even the *strong* solution notion of Definition 2.32(i), which is the SPDE analogue closest to a classical PDE solution, requiring $X(t) \in \text{dom}(A)$, is therefore an integral-equation notion, not a pointwise-differential one. This is what distinguishes it from an actual classical solution and is the reason all solution concepts in this thesis are stated as integral identities.

Definition 2.31 (Martingale). Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and K a Banach space. An $\mathbb{F}$ -adapted, integrable process $M: [0, T] \times \Omega \rightarrow K$ is called an $\mathbb{F}$ -*martingale* if $\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$ $\mathbb{P}$ -a.s. for all $0 \leq s \leq t \leq T$. It is called a *local martingale* if there exists a sequence of $\mathbb{F}$ -stopping times $\tau_n \uparrow T$ $\mathbb{P}$ -a.s. such that each stopped process $M(\cdot \wedge \tau_n)$ is an $\mathbb{F}$ -martingale. Every martingale is a local martingale, but a local martingale need only be a genuine martingale up to an “escape to infinity” along (τ_n) , which is why local martingales without further integrability information cannot in general be handled by taking expectations directly. All stochastic integrals $\int_0^\cdot \Phi dW$ of Section 2.2 are, without further assumptions, local martingales. They are genuine martingales precisely under the square-integrability hypothesis $\mathbb{E}[\int_0^T \|\Phi(s)\|_{\text{HS}}^2 ds] < \infty$ used there.

For a real-valued, continuous local martingale M with $M(0) = 0$, the *quadratic variation* $\langle M \rangle = (\langle M \rangle_t)_{t \in [0, T]}$ is the (up to indistinguishability) unique continuous, non-decreasing, $\mathbb{F}$ -adapted process with $\langle M \rangle_0 = 0$ such that $M^2 - \langle M \rangle$ is a local martingale (Doob–Meyer decomposition; see [DZ14]). More generally, for a continuous, K -valued (or $L_2(\mathcal{H}, K)$ -valued) local martingale M arising as a stochastic integral $M(t) = \int_0^t \Phi(s) dW(s)$ as in Section 2.2, we write $\langle M \rangle_t := \int_0^t \|\Phi(s)\|_{\text{HS}}^2 ds$ for the (real-valued, scalar) quadratic variation of M , so that $\|M(t)\|_K^2 - \langle M \rangle_t$ is a real-valued local martingale. This is the quantity controlled by the Burkholder–Davis–Gundy inequality of Theorem 2.35 below.

Definition 2.32 (Solution concepts). Let $X_0 \in K$ be $\mathcal{F}_0$ -measurable and let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space carrying a cylindrical Wiener process W on $\mathcal{H}$ over $\mathfrak{A}$.

- (i) An $\mathbb{F}$ -progressively measurable process X with $X(t) \in \text{dom}(A)$ for a.e. t , $\mathbb{P}$ -a.s., is a *strong solution* of (2.19) if $\mathbb{P}$ -a.s.

$$X(t) = X_0 + \int_0^t (AX(s) + F(X(s))) ds + \int_0^t G(X(s)) dW(s) \quad \text{for all } t \in [0, T].$$

- (ii) X is a *weak solution* if for every $\zeta \in \text{dom}(A^*)$ and $t \in [0, T]$, $\mathbb{P}$ -a.s.

$$\langle X(t), \zeta \rangle_K = \langle X_0, \zeta \rangle_K + \int_0^t \left(\langle X(s), A^* \zeta \rangle_K + \langle F(X(s)), \zeta \rangle_K \right) ds + \int_0^t \langle G(X(s)), \zeta \rangle_K dW(s).$$

(iii) X is a *mild solution* if $\mathbb{P}$ -a.s.

$$X(t) = e^{tA}X_0 + \int_0^t e^{(t-s)A}F(X(s)) \, ds + \int_0^t e^{(t-s)A}G(X(s)) \, d\mathcal{W}(s) \quad \text{for all } t \in [0, T].$$

(iv) X is a *martingale solution* if, instead of being defined on a prescribed $\mathfrak{A}$, one only requires the existence of *some* filtered probability space $\mathfrak{A}$ carrying a cylindrical Wiener process $\mathcal{W}$ over $\mathfrak{A}$ and a process X that is a solution (in one of the senses above) of (2.19) over $\mathfrak{A}$; i.e. only the joint law of $(X, \mathcal{W})$, not the underlying probability space itself, is prescribed.

For the sub-steps of the construction in Chapter 4 where the equation is not merely Lipschitz but genuinely *linear* (the drift affine in X via a fixed forcing f , and the noise coefficient a fixed bounded linear operator, rather than a Lipschitz nonlinearity of X) [DZ14] treats

$$dX(t) = (AX(t) + f(t)) \, dt + G(X(t)) \, d\mathcal{W}(t), \quad X(0) = X_0, \quad (2.20)$$

separately (Chapter 6), where A generates a C_0 -semigroup on H , X_0 is an $\mathcal{F}_0$ -measurable H -valued random variable, f is a predictable process with a.s. locally integrable trajectories, and $G: \text{dom}(G) \subseteq H \rightarrow L_2(\mathcal{H}, H)$ is a linear operator with $G(x)$ Hilbert–Schmidt for every $x \in \text{dom}(G)$.

Theorem 2.33 ([DZ14, Theorem 6.7]). *Assume A generates a C_0 -semigroup on H , $\mathbb{E}\|X_0\|_H^2 < \infty$, and $G \in L(H; L_2(\mathcal{H}, H))$ is a bounded linear operator. Then (2.20) has a unique mild solution $X \in \mathcal{N}_W^2(0, T; H)$, the space of H -valued predictable processes with $\|X\|_{\mathcal{N}_W^2(0, T; H)}^2 := \sup_{t \in [0, T]} \mathbb{E}\|X(t)\|_H^2 < \infty$ (weaker than $\mathcal{M}_{\mathfrak{A}}^2(0, T; H)$ of Section 2.2, where the supremum is taken inside the expectation), which coincides with the weak solution.*

Every strong solution is a weak solution (test (2.19) against $\zeta \in \text{dom}(A^*)$), and every strong solution is a mild solution (apply the variation-of-constants formula, i.e. Itô’s formula to $t \mapsto e^{(t-s)A}X(s)$, cf. Theorem 2.37). Under mild additional assumptions (e.g. $X_0 \in \text{dom}(A)$ and enough regularity of A , F , G for the stochastic convolution to again take values in $\text{dom}(A)$), a mild solution is also a weak solution, and the three notions coincide whenever X is regular enough for all integrals above to make sense classically; see [DZ14] for the precise statements. Since $\text{dom}(A) = \{w \in H^2(\mathcal{O}) : \partial_\nu w|_{\partial\mathcal{O}} = 0\}$ is a strictly smaller space than $L^2(\mathcal{O})$, and the truncated solutions constructed in Section 4.2.2 are not a priori known to take values there, it is the *mild* formulation that is used throughout this thesis.

2.4 The Burkholder–Davis–Gundy Inequality

Throughout this thesis, stochastic integrals are taken against a cylindrical Wiener process on the Hilbert space $\mathcal{H}$, with $L_2(\mathcal{H}, L^2(\mathcal{O}))$ -valued (Hilbert–Schmidt) integrands, in the sense of [DZ14, Section 4.1]. The following maximal inequality, in the specific form used throughout Chapter 4 and matching the version used in [HMT22a], controls moments of the running supremum of such integrals in terms of moments of the quadratic variation.

Theorem 2.34 (Kolmogorov–Chentsov). *Let Y be a Banach space and let $X: [0, T] \times \Omega \rightarrow Y$ be a stochastic process such that for some $p > 0$, $\alpha > 0$ and $C > 0$,*

$$\mathbb{E}[\|X(t) - X(s)\|_Y^p] \leq C|t - s|^{1+\alpha} \quad \text{for all } s, t \in [0, T].$$

Then there exists a modification $\tilde{X}$ of X which has $\mathbb{P}$ -a.s. Hölder-continuous paths with exponent $\beta \in (0, \frac{\alpha}{p})$, and in particular

$$\mathbb{E} \left[\sup_{|t-s| \leq \delta} \|\tilde{X}(t) - \tilde{X}(s)\|_Y^p \right] \leq C' \delta^\alpha \quad \text{for all } \delta > 0.$$

Proof. See [DZ14, Theorem 3.3]. □

Theorem 2.35 (Burkholder–Davis–Gundy inequality). *Let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, $\mathcal{W}$ a cylindrical Wiener process on $\mathcal{H}$ over $\mathfrak{A}$, and $\Phi \in \mathcal{M}_{\mathfrak{A}}^2(0, T; L_2(\mathcal{H}, L^2(\mathcal{O})))$. Then for every $p \geq 2$ there exists a constant $C = C(p) > 0$, independent of Φ and T , such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left\| \int_0^t \Phi(s) d\mathcal{W}(s) \right\|_{L^2}^p \right] \leq C \mathbb{E} \left[\left(\int_0^T \|\Phi(s)\|_{L_2(\mathcal{H}, L^2)}^2 ds \right)^{p/2} \right]. \quad (2.21)$$

Proof. See [DZ14]. □

Lemma 2.36 (Burkholder–Davis–Gundy, localised). *Under the assumptions of Theorem 2.35, for any $0 \leq s \leq T$ and $p \geq 2$,*

$$\mathbb{E}_n \left[\sup_{t \in [s, s+h]} \left\| \int_s^t \Phi(\sigma) d\mathcal{W}(\sigma) \right\|_{L^2}^p \right] \leq C \mathbb{E}_n \left[\left(\int_s^{s+h} \|\Phi(\sigma)\|_{L_2}^2 d\sigma \right)^{p/2} \right] \quad (2.22)$$

for all $h > 0$ with $s + h \leq T$, where $C = C(p) > 0$ is the same constant as in Theorem 2.35.

Proof. Set $\tilde{\Phi}(r) := \mathbf{1}_{[s, s+h]}(r) \Phi(r)$ and apply Theorem 2.35 from 0 to $s + h$. □

2.5 Stochastic Convolutions, Itô's Formula and the Yosida Approximation

For an analytic semigroup $(e^{tA})_{t \geq 0}$ generated by an operator A (in our applications $A = \Delta$ or $A = \Delta + \gamma I$ with Neumann boundary conditions, cf. Lemma 2.19) and a progressively measurable, $L_2(\mathcal{H}, L^2(\mathcal{O}))$ -valued integrand Φ , the process

$$(\mathfrak{S}_A \Phi)(t) := \int_0^t e^{(t-s)A} \Phi(s) d\mathcal{W}(s), \quad t \in [0, T],$$

is called a *stochastic convolution*. Unlike the stochastic integral of Section 2.2, $\mathfrak{S}_A \Phi$ is in general not a martingale, since the semigroup factor $e^{(t-s)A}$ depends on t .

Theorem 2.37 (Itô formula). *Let H be a Hilbert space, $\mathcal{W}$ a cylindrical Wiener-process on $\mathcal{H}$ over $\mathfrak{A}$ and let $\phi \in C^2(H; \mathbb{R})$. Let $X \in \mathcal{M}_{\mathfrak{A}}^2(0, T; H)$ be an adapted process which fulfills*

$$X(t) = X(0) + \int_0^t F(s) ds + \int_0^t G(s) d\mathcal{W}(s), \quad \forall t \in [0, T],$$

where $F \in L^1(0, T; H)$ and $G \in L^2(0, T; L_2(\mathcal{H}, H))$ $\mathbb{P}$ -a.s. Then for all $t \in [0, T]$ the following holds $\mathbb{P}$ -a.s.

$$\begin{aligned} \phi(X(t)) &= \phi(X(0)) + \int_0^t D\phi(X(s))[F(s)] ds + \int_0^t D\phi(X(s))[G(s) d\mathcal{W}(s)] \\ &\quad + \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_0^t D^2\phi(X(s))[G(s)e_k, G(s)e_k] ds. \end{aligned}$$

Proof. See [DZ14, Theorem 4.32]. □

Definition 2.38 (Yosida Approximation). Let H be a Hilbert space and $A: \text{dom}(A) \subseteq H \rightarrow H$ be densely defined, positive and self-adjoint (possibly unbounded). We define the Yosida Approximation

$$Y_\varepsilon = Y_{\varepsilon, A} := (I + \varepsilon A)^{-1}: H \rightarrow \text{dom}(A), \quad \varepsilon \in (0, 1).$$

Lemma 2.39. *For H and A as before the Yosida Approximation fulfills the following:*

- (i) $Y_\varepsilon w \in \text{dom}(A) \quad \forall w \in H \quad \text{and} \quad AY_\varepsilon w = Y_\varepsilon Aw \quad \forall w \in \text{dom}(A),$
- (ii) $Y_\varepsilon A = \frac{1}{\varepsilon}(I - Y_\varepsilon), \quad \|Y_k\|_{\text{op}} \leq 1 \quad \text{and} \quad \|Y_k A\|_{\text{op}} \leq \frac{1}{\varepsilon},$
- (iii) $w = \lim_{\varepsilon \rightarrow 0} Y_\varepsilon w \quad \forall w \in H \quad \text{and} \quad Aw = \lim_{\varepsilon \rightarrow 0} AY_\varepsilon w \quad \forall w \in \text{dom}(A).$

Proof. (i) Let $w \in H$ and set $v := Y_\varepsilon w \in \text{dom}(A) \subseteq H$, so $w = (I + \varepsilon A)v = v + \varepsilon Av$. This shows $Av = \frac{1}{\varepsilon}(w - v) \in H$, hence $v = Y_\varepsilon w \in \text{dom}(A)$. Additionally on $\text{dom}(A)$

$$AY_\varepsilon = \frac{1}{\varepsilon}(I + \varepsilon A - I)(I + \varepsilon A)^{-1} = \frac{1}{\varepsilon}(I - (I + \varepsilon A)^{-1}) = \frac{1}{\varepsilon}(I + \varepsilon A)^{-1}(I + \varepsilon A - I) = Y_\varepsilon A.$$

(ii) The identity follows from (i). For $\|Y_\varepsilon\|_{\text{op}} \leq 1$: let $w \in H$ and $v := Y_\varepsilon w \in \text{dom}(A)$. Since A is positive:

$$\|w\|_H \|v\|_H \geq \langle w, v \rangle_H = \langle (I + \varepsilon A)v, v \rangle_H = \|v\|_H^2 + \varepsilon \langle Av, v \rangle_H \geq \|v\|_H^2,$$

hence $\|Y_\varepsilon w\|_H \leq \langle w, Y_\varepsilon w \rangle_H \leq \|w\|_H$. For $\|Y_\varepsilon A\|_{\text{op}} \leq \frac{1}{\varepsilon}$: from the above,

$$\|w - Y_\varepsilon w\|_H^2 = \|w\|_H^2 - 2 \langle w, Y_\varepsilon w \rangle_H + \|Y_\varepsilon w\|_H^2 \leq \|w\|_H^2 - \|Y_\varepsilon w\|_H^2 \leq \|w\|_H^2.$$

Hence by the identity:

$$\|Y_\varepsilon Aw\|_H = \frac{1}{\varepsilon} \|w - Y_\varepsilon w\|_H \leq \frac{1}{\varepsilon} \|w\|_H.$$

(iii) For $w \in \text{dom}(A)$: using the identity and $\|Y_\varepsilon\|_{\text{op}} \leq 1$:

$$\|Y_\varepsilon w - w\|_H = \varepsilon \|Y_\varepsilon A w\|_H \leq \varepsilon \|A w\|_H \xrightarrow{\varepsilon \rightarrow 0} 0.$$

For general $w \in H$: since $\text{dom}(A)$ is dense in H , for any $\eta > 0$ choose $\tilde{w} \in \text{dom}(A)$ with $\|w - \tilde{w}\|_H < \eta$. Then:

$$\|Y_\varepsilon w - w\|_H \leq \|Y_\varepsilon(w - \tilde{w})\|_H + \|\tilde{w} - w\|_H + \|Y_\varepsilon \tilde{w} - \tilde{w}\|_H \leq 2\eta + \varepsilon \|A \tilde{w}\|_H.$$

For ε small enough the right-hand side is less than 3η , and since $\eta > 0$ was arbitrary, $Y_\varepsilon w \rightarrow w$.

For $w \in \text{dom}(A)$: by (i), $A Y_\varepsilon w = Y_\varepsilon A w$, and by the first part applied to $A w \in H$: $Y_\varepsilon(A w) \rightarrow A w$. $\square$

Lemma 2.40. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $Y_\varepsilon = (I - \varepsilon \Delta_N)^{-1} : L^2(\mathcal{O}) \rightarrow \text{dom}(-\Delta_N) = \{w \in H^2 \mid \partial_\nu w|_{\partial\mathcal{O}} = 0\}$ be the Yosida Approximation of the Laplacian with Neumann boundary condition. Then Y_ε maintains non negativity and the Neumann boundary condition, i.e., for $w \in L^2(\mathcal{O})$ we have $\partial_\nu(Y_\varepsilon w)|_{\partial\mathcal{O}} = 0$ and*

$$w \geq 0 \text{ a.e.} \implies Y_\varepsilon w \geq 0 \text{ a.e.}$$

Proof. Let $w \in L^2(\mathcal{O})$. Then $u := Y_\varepsilon w \in \text{dom}(-\Delta_N)$, hence $\partial_\nu u|_{\partial\mathcal{O}} = 0$. Let now also $w \geq 0$ a.e. We obtain that u solves

$$\int_{\mathcal{O}} u \varphi + \varepsilon \int_{\mathcal{O}} \nabla u \nabla \varphi = \int_{\mathcal{O}} w \varphi \quad \forall \varphi \in H^1(\mathcal{O}).$$

If we now test with $\varphi = u^- = \max(-u, 0) \in H^1(\mathcal{O})$ with $\nabla u^- = -\nabla u \cdot \mathbf{1}_{u < 0}$ we obtain $u u^- = -|u^-|^2$ and $\nabla u \nabla u^- = -|\nabla u^-|^2$ and hence

$$-\int_{\mathcal{O}} |u^-|^2 - \varepsilon \int_{\mathcal{O}} |\nabla u^-|^2 = \int_{\mathcal{O}} w u^- \geq 0,$$

which implies $\|u^-\|_{L^2_{\mathcal{O}}} = 0$, i.e., $u^- = 0$ a.e., hence $u \geq 0$ a.e. $\square$

2.6 Stratonovich Versus Itô

For a sufficiently regular real-valued process X and a real Wiener process β , the *Stratonovich integral* $\int_0^t X(s) \circ d\beta(s)$ is defined as the limit in probability of symmetric (midpoint) Riemann sums

$$\sum_i \frac{1}{2} (X(t_{i+1}) + X(t_i)) (\beta(t_{i+1}) - \beta(t_i))$$

over partitions of $[0, t]$ with mesh tending to zero, as opposed to the Itô integral, which uses the left endpoint $X(t_i)$ instead of the midpoint average. When $X = f(\beta)$ for a C^2 function f , or more generally when X solves a stochastic differential equation of the same type, the two integrals differ by an explicit correction term, and a Stratonovich SDE $dX =$

$b(X) dt + \sigma(X) \circ d\beta$ is equivalent to the Itô SDE $dX = (b(X) + \frac{1}{2}\sigma(X)\sigma'(X)) dt + \sigma(X) d\beta$; see [DW14].

As discussed in Section 1.4, the Wong–Zakai theorem [WZ65] identifies the Stratonovich integral as the natural limit of physically realizable (smoothly correlated) noise approximations, which is why it is the appropriate interpretation for the intrinsic multiplicative noise $u \circ dW$ in the cell-density equation of (4.1). Since the subsequent analysis (Itô’s formula, martingale arguments, the Burkholder–Davis–Gundy inequality) is most conveniently carried out in the Itô calculus, the Stratonovich system (4.1) will be converted into an equivalent Itô system before the fixed-point construction.

2.7 Compactness Methods and the Schauder–Tychonoff Fixed Point

This section collects the compactness tools used to pass from the approximating solutions $\mathfrak{T}_n\xi$ of Section 4.2.2 to an actual mild martingale solution, since $\mathfrak{T}_n$ is not known to be a compact operator (Section 2.7) and only tightness of the laws of $\mathfrak{T}_n\xi$ can be established.

A metric space is called *Polish* if it is separable and complete. This is the standing regularity assumption for the spaces of measures and processes below, and is satisfied by the path space $C^0([0, T]; L^2(\mathcal{O}))$ used throughout Chapter 4.

Definition 2.41. For a complete metric space X by $\mathcal{M}(X)$ we denote the space of all measures defined on $\mathcal{B}_X$, i.e., all non-negative, countably additive functions μ on $\mathcal{B}_X$ with $\mu(X) = 1$.

Definition 2.42. For a complete metric space X the family $\Gamma \subseteq \mathcal{M}(X)$ is called uniformly tight if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq X$ such that $\mu(K_\varepsilon) \geq 1 - \varepsilon$ for all $\mu \in \Gamma$.

Theorem 2.43 (Prokhorov). *Let X be a complete separable metric space and $\Gamma \subseteq \mathcal{M}(X)$. Then*

$$\bar{\Gamma} \text{ is compact } \iff \Gamma \text{ is uniformly tight.}$$

Proof. The proof can be found in [Par67]. □

Let $T > 0$, Y be a Banach space and $\omega_x(\delta) := \sup_{|t-s| \leq \delta} \|x(t) - x(s)\|_Y$.

Lemma 2.44 (Arzelà–Ascoli). *Let $A \subseteq C^0([0, T]; Y)$. Then $\bar{A}$ is compact if and only if the following two conditions are satisfied:*

$$\overline{\{x(0) : x \in A\}} \text{ is compact in } Y \quad \text{and} \quad \lim_{\delta \searrow 0} \sup_{x \in A} \omega_x(\delta) = 0.$$

Proof. Immediate consequence of the Ascoli theorem (see [Kel61, pp. 233–234]) and the compactness of the interval $[0, T]$. □

Theorem 2.45. *Let Γ be a set of probability measures on $X = C^0([0, T]; Y)$. Then $\bar{\Gamma}$ is compact if and only if for each $\varepsilon > 0$ there exists a compact set $L_\varepsilon \subset Y$ and a function $\omega(\delta, \varepsilon)$*

which decreases to zero as $\delta \searrow 0$ such that for all $\mu \in \Gamma$ the following two conditions are satisfied:

$$\mu(\{x \in X : x(0) \in L_\varepsilon\}) > 1 - \frac{\varepsilon}{2} \quad \text{and} \quad \mu(\{x \in X : \omega_x(\delta) \leq \omega(\delta, \varepsilon) \forall \delta\}) > 1 - \frac{\varepsilon}{2}.$$

Proof. $\Leftarrow$: Suppose that Γ satisfies the two conditions. For $\varepsilon > 0$ the set $K_\varepsilon = A_\varepsilon \cap B_\varepsilon$ is closed, where

$$A_\varepsilon := \{x \in X : x(0) \in L_\varepsilon\} \quad \text{and} \quad B_\varepsilon := \{x \in X : \omega_x(\delta) \leq \omega(\delta, \varepsilon) \forall \delta\}.$$

By Lemma 2.44, $K_\varepsilon = \overline{K_\varepsilon}$ is compact and for all $\mu \in \Gamma$ we have

$$\mu(K_\varepsilon^c) = \mu(A_\varepsilon^c \cup B_\varepsilon^c) \leq \mu(A_\varepsilon^c) + \mu(B_\varepsilon^c) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \implies \mu(K_\varepsilon) > 1 - \varepsilon.$$

Hence by Theorem 2.43, $\overline{\Gamma}$ is compact.

$\implies$: Suppose that $\overline{\Gamma}$ is compact. Then by Theorem 2.43, there exists a compact set K_ε such that $\mu(K_\varepsilon) > 1 - \frac{\varepsilon}{2}$ for all $\mu \in \Gamma$. Set

$$L_\varepsilon := \overline{\{x(0) : x \in K_\varepsilon\}} \subset Y \quad \text{and} \quad \omega(\delta, \varepsilon) = \sup_{x \in K_\varepsilon} \omega_x(\delta).$$

Since evaluation at $t = 0$ is continuous on X , L_ε is compact in Y as the closure of the image of the compact set K_ε . We have $\omega(\delta, \varepsilon) \searrow 0$ as $\delta \searrow 0$ by Lemma 2.44. Since for all $x \in K_\varepsilon$ we have $x(0) \in L_\varepsilon$ and $\omega_x(\delta) \leq \omega(\delta, \varepsilon)$,

$$\begin{aligned} \mu(\{x \in X : x(0) \in L_\varepsilon\}) &\geq \mu(K_\varepsilon) > 1 - \frac{\varepsilon}{2}, \\ \mu(\{x \in X : \omega_x(\delta) \leq \omega(\delta, \varepsilon) \forall \delta\}) &\geq \mu(K_\varepsilon) > 1 - \frac{\varepsilon}{2}, \end{aligned}$$

for all $\mu \in \Gamma$. □

Tightness alone does not give a fixed point, since weak convergence of laws does not give almost-sure convergence of the underlying random variables on a fixed probability space. This is remedied by the following representation theorem.

Theorem 2.46 (Skorokhod representation). *Let S be a Polish space and $(\mu_n)_{n \in \mathbb{N}}$, μ probability measures on S with $\mu_n \rightharpoonup \mu$. Then there exist S -valued random variables $\tilde{X}_n$, $\tilde{X}$, $n \in \mathbb{N}$, on a common probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, with $\text{Law}(\tilde{X}_n) = \mu_n$, $\text{Law}(\tilde{X}) = \mu$, such that $\tilde{X}_n \rightarrow \tilde{X}$ $\tilde{\mathbb{P}}$ -a.s.*

Proof. See [Par67]. □

The Banach and Schauder fixed-point theorems of Section 2.1 (Theorems 2.28, 2.29) require the underlying space to be a genuine Banach space and the relevant set to be compact, or the map itself to be compact. Tychonoff's theorem extends the Schauder fixed-point principle to the setting of a general locally convex topological vector space X instead of a Banach space. This *Schauder–Tychonoff* theorem is the abstract principle underlying the fixed-point construction used in this thesis.

3 The Deterministic Consumption Model

3.1 Formulation and Basic Properties

Let $d \in \{1, 2, 3\}$ and $\mathcal{O} \subset \mathbb{R}^d$ a bounded, convex domain with C^2 boundary (for $d = 1$, i.e. a bounded interval, both conditions hold automatically). This is the standing hypothesis of [Win12; TW12], on which the global existence theory of this chapter (Theorem 3.15) relies, and it matches the hypotheses of Lemma 2.20 and Lemma 2.21 used within this chapter; mere Lipschitz regularity, sufficient for Chapter 2, is not enough here.

We consider the Keller–Segel chemotaxis-consumption system

$$\begin{cases} \partial_t u = \Delta u - \operatorname{div}(u \nabla v), & \text{in } \mathcal{O} \times (0, \infty), \\ \partial_t v = \Delta v - uv, & \text{in } \mathcal{O} \times (0, \infty), \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\mathcal{O} \times (0, \infty), \\ (u(0), v(0)) = (u_0, v_0), & \text{in } \mathcal{O}. \end{cases} \quad (3.1)$$

The homogeneous boundary conditions of this system reflect the no-flux boundary conditions $(\nabla u - u \nabla v) \cdot \nu = 0$ and $\nabla v \cdot \nu = 0$ on $\partial\mathcal{O}$, which ensure that neither cells nor chemoattractant cross the boundary, and which are the natural choice for this system. The initial data are required to satisfy

$$\begin{cases} u_0 \in C(\bar{\mathcal{O}}), & u_0 > 0 \text{ in } \bar{\mathcal{O}}, \\ v_0 \in W^{1,\infty}(\mathcal{O}), & v_0 > 0 \text{ in } \bar{\mathcal{O}}. \end{cases} \quad (3.2)$$

Definition 3.1 (Weak solutions). The pair (u, v) with

$$u \in L^1_{\text{loc}}([0, \infty); L^1(\mathcal{O})) \quad \text{and} \quad v \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\mathcal{O})) \quad (3.3)$$

is called a global weak solution of (3.1) if

$$uv, u \nabla v \in L^1_{\text{loc}}([0, \infty); L^1(\mathcal{O})) \quad (3.4)$$

and for all $\varphi \in C_0^\infty(\bar{\mathcal{O}} \times [0, \infty))$ the following two identities hold

$$-\int_0^\infty \int_{\mathcal{O}} u \varphi_t - \int_{\mathcal{O}} u_0 \varphi(\cdot, 0) = -\int_0^\infty \int_{\mathcal{O}} \nabla u \cdot \nabla \varphi + \int_0^\infty \int_{\mathcal{O}} u \nabla v \cdot \nabla \varphi \quad \text{and} \quad (3.5)$$

$$-\int_0^\infty \int_{\mathcal{O}} v \varphi_t - \int_{\mathcal{O}} v_0 \varphi(\cdot, 0) = -\int_0^\infty \int_{\mathcal{O}} \nabla v \cdot \nabla \varphi - \int_0^\infty \int_{\mathcal{O}} uv \varphi. \quad (3.6)$$

Remark 3.2 (Existence and regularity by dimension). The type of global solution that exists depends on the spatial dimension d :

- $d = 1$: global classical solution by standard parabolic theory.
- $d = 2$: global classical solution in bounded convex domains, see [Win12, Theorem 1.1 i)].
- $d = 3$: global weak solution for arbitrarily large initial data, see [Win12, Theorem 1.1 ii)]; eventual smoothness for convex $\mathcal{O}$ with smooth boundary is established in [TW12].

In all cases the regularized approximation and the properties of Lemma 3.5 are dimension-independent.

Definition 3.3 (Regularized Problem and strong solutions). For $F_\varepsilon(s) = \frac{1}{\varepsilon} \log(1 + \varepsilon s)$ we define the system

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \operatorname{div}(u_\varepsilon F'_\varepsilon(u_\varepsilon) \nabla v_\varepsilon), & \text{in } \mathcal{O} \times (0, \infty), \\ \partial_t v_\varepsilon = \Delta v_\varepsilon - F_\varepsilon(u_\varepsilon) v_\varepsilon, & \text{in } \mathcal{O} \times (0, \infty), \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & \text{on } \partial \mathcal{O} \times (0, \infty), \\ (u_\varepsilon(0), v_\varepsilon(0)) = (u_0, v_0), & \text{in } \mathcal{O}. \end{cases} \quad (3.7)$$

For each $\varepsilon \in (0, 1)$, problem (3.7) possesses a unique global classical solution $(u_\varepsilon, v_\varepsilon) \in C^0(\bar{\mathcal{O}} \times [0, \infty)) \cap C^{2,1}(\bar{\mathcal{O}} \times (0, \infty))$ satisfying $u_\varepsilon, v_\varepsilon > 0$ in $\bar{\mathcal{O}} \times [0, \infty)$, see [TW12, Lemma 2.1].

Remark 3.4. Note that we have

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(s) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \log(1 + \varepsilon s) = s \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} F'_\varepsilon(s) = \lim_{\varepsilon \rightarrow 0} \frac{1}{1 + \varepsilon s} = 1 \quad (3.8)$$

for all $s \geq 0$.

Lemma 3.5 (Nonnegativity and L^∞ -bound for v). *The global solution (u, v) of (3.1) satisfies*

$$u, v \geq 0 \quad \text{a.e. in } \mathcal{O} \times (0, \infty), \quad (3.9)$$

and

$$\|v(\cdot, t)\|_{L^\infty(\mathcal{O})} \leq \|v_0\|_{L^\infty(\mathcal{O})} \quad \text{for all } t \geq 0. \quad (3.10)$$

Proof. The classical solutions $(u_\varepsilon, v_\varepsilon)$ of (3.7) satisfy $u_\varepsilon, v_\varepsilon > 0$ pointwise, so nonnegativity of u and v follows upon passing $\varepsilon = \varepsilon_j \rightarrow 0$ via a.e. convergence. For the L^∞ bound: since $F_\varepsilon(u_\varepsilon) \geq 0$ and $v_\varepsilon \geq 0$, the second equation in (3.7) yields

$$v_{\varepsilon t} \leq \Delta v_\varepsilon,$$

i.e., v_ε is a subsolution of the heat equation. The parabolic maximum principle then implies that $t \mapsto \|v_\varepsilon(\cdot, t)\|_{L^\infty(\mathcal{O})}$ is nonincreasing, giving $\|v_\varepsilon(\cdot, t)\|_{L^\infty} \leq \|v_0\|_{L^\infty}$ uniformly in ε . Passing $\varepsilon_j \rightarrow 0$ yields (3.10); cf. [TW12, Lemma 2.3]. $\square$

Lemma 3.6 (Conservation of mass). *Let (u, v) be a sufficiently regular solution to (3.1). Then the total cell mass is conserved, i.e.,*

$$\int_{\mathcal{O}} u(x, t) \, dx = \int_{\mathcal{O}} u_0(x) \, dx \quad \text{for all } t \in (0, \infty). \quad (3.11)$$

Proof. For sufficiently regular solutions, differentiation under the integral sign is justified by the dominated convergence theorem, and we compute

$$\frac{d}{dt} \int_{\mathcal{O}} u = \int_{\mathcal{O}} u_t = \int_{\mathcal{O}} \operatorname{div}(\nabla u - u \nabla v) = \int_{\partial \mathcal{O}} (\nabla u - u \nabla v) \cdot \nu = 0, \quad (3.12)$$

using the equation (3.1), the divergence theorem, and the no-flux boundary conditions. $\square$

Remark 3.7. For weak solutions in the sense of Definition 3.1, the same identity follows formally by choosing $\varphi \equiv 1$ as test function in (3.5). A rigorous justification requires an approximation argument, see e.g. [TW12, Lemma 2.2].

Lemma 3.8 (Dissipation of $\int v$).

$$\|v(\cdot, t)\|_{L^1(\mathcal{O})} \leq \|v_0\|_{L^1(\mathcal{O})} \quad \text{for all } t \geq 0. \quad (3.13)$$

Proof. Since $u, v \geq 0$ by Lemma 3.5, integrating the v -equation of (3.1) over $\mathcal{O}$ and using the no-flux boundary condition gives $\frac{d}{dt} \int_{\mathcal{O}} v = \int_{\mathcal{O}} \Delta v - \int_{\mathcal{O}} uv = - \int_{\mathcal{O}} uv \leq 0$, so $t \mapsto \|v(\cdot, t)\|_{L^1(\mathcal{O})} = \int_{\mathcal{O}} v(\cdot, t)$ is non-increasing. $\square$

3.2 Local Existence and Uniqueness

For the sake of completeness, we recall the standard local well-posedness result for system (3.1), which follows from the analytic semigroup framework via a Banach fixed-point argument, by the same technique used for the regularized system in the next section (there yielding *global* existence, see [TW12, Lemma 2.1]); see [Paz83, Chapter 7] for the underlying abstract semigroup local-existence theory.

Theorem 3.9 (Local existence and uniqueness). *Let u_0, v_0 satisfy the regularity assumptions stated above. Then there exist $T_{\max} \in (0, \infty]$ and a unique pair of classical solutions*

$$(u, v) \in C^0(\bar{\mathcal{O}} \times [0, T_{\max})) \cap C^{2,1}(\bar{\mathcal{O}} \times (0, T_{\max}))$$

to (3.1), and either $T_{\max} = \infty$ or $\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \rightarrow \infty$ as $t \rightarrow T_{\max}$.

3.3 Global Existence via Energy Estimates

The local existence result of Theorem 3.9 does not exclude blow-up in finite time. To rule this out, we derive uniform a priori bounds via an energy functional adapted to the consumption structure of (3.1).

Definition 3.10 (Energy functional). For non-negative $u \in L \log L(\mathcal{O})$ and $v \in W^{1,2}(\mathcal{O})$ with $v > 0$ a.e. in $\mathcal{O}$, define the energy functional

$$W(u, v) := \int_{\mathcal{O}} u \log u + 2 \int_{\mathcal{O}} |\nabla \sqrt{v}|^2. \quad (3.14)$$

Remark 3.11. The second term in (3.14) is the Fisher information of v and equals $\frac{1}{2} \int_{\mathcal{O}} \frac{|\nabla v|^2}{v}$. It is the natural counterpart to the entropy $\int_{\mathcal{O}} u \log u$ along the flow of (3.1), in contrast to the classical production model where a cross term $-\rho \int_{\mathcal{O}} uv$ is required to compensate the destabilizing source $+\beta u$ in the equation for v .

Lemma 3.12. Let $d \in \{2, 3\}$. Then the solution $(u_\varepsilon, v_\varepsilon)$ of (3.7) satisfies the identity

$$\frac{d}{dt} W(u_\varepsilon, v_\varepsilon) + \int_{\mathcal{O}} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_{\mathcal{O}} v_\varepsilon |D^2 \log v_\varepsilon|^2 = -\frac{1}{2} \int_{\mathcal{O}} F_\varepsilon(u_\varepsilon) \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{1}{2} \int_{\partial \mathcal{O}} \frac{1}{v_\varepsilon} \partial_\nu |\nabla v_\varepsilon|^2 \quad (3.15)$$

Proof. This is the general chemotaxis-fluid energy identity of [Win12, Section 3], specialised to the present setting $(n, c, u, P) = (u_\varepsilon, v_\varepsilon, 0, 0)$ with $f = \text{id}$, $\chi \equiv 1$ (hence $g = \text{id}$), $\psi(s) = 2\sqrt{s}$ and $\rho(s) = \log s$ in the notation of that paper; the term $\int_{\mathcal{O}} v_\varepsilon |D^2 \log v_\varepsilon|^2$ is precisely identity (2.18) of Lemma 2.21, applied with $w = v_\varepsilon$, combined with the entropy identity for the u_ε -equation. We do not reproduce the (lengthy) derivation here and instead refer to the cited source. $\square$

Lemma 3.13 (Energy–dissipation inequality). Let (u, v) be a sufficiently regular non-negative solution to (3.1) with $v > 0$. Then

$$\frac{d}{dt} W(u, v) + \mathcal{D}(u, v) \leq 0, \quad (3.16)$$

where the dissipation $\mathcal{D} \geq 0$ is defined as

$$\mathcal{D}(u, v) = \int_{\mathcal{O}} \frac{|\nabla u|^2}{u} + \int_{\mathcal{O}} v |D^2 \log v|^2 + \frac{1}{2} \int_{\mathcal{O}} u \frac{|\nabla v|^2}{v}. \quad (3.17)$$

Sketch. For the regularized system (3.7), defining $\mathcal{D}_\varepsilon$ analogously to (3.17) with u replaced by $F_\varepsilon(u)$ in the last integral, Lemma 3.12 together with Lemma 2.20 gives $\frac{d}{dt} W(u_\varepsilon, v_\varepsilon) + \mathcal{D}_\varepsilon(u_\varepsilon, v_\varepsilon) = \frac{1}{2} \int_{\partial \mathcal{O}} \frac{1}{v_\varepsilon} \partial_\nu |\nabla v_\varepsilon|^2 \leq 0$. Since $F_\varepsilon \rightarrow \text{id}$, $F'_\varepsilon \rightarrow 1$ locally uniformly as $\varepsilon \rightarrow 0$, and $(u_\varepsilon, v_\varepsilon) \rightarrow (u, v)$ with the necessary gradient/Hessian bounds on compact time intervals by the a priori estimates underlying [TW12], the claim follows upon passing to the limit $\varepsilon \rightarrow 0$; we omit the routine details. $\square$

Corollary 3.14 (Uniform energy bound). Under the assumptions of Lemma 3.13,

$$W(u(\cdot, t), v(\cdot, t)) \leq W(u_0, v_0) \quad \text{for all } t \in [0, T_{\max}), \quad (3.18)$$

and consequently

$$\sup_{t \in [0, T_{\max})} \|u(\cdot, t)\|_{L \log L(\mathcal{O})} + \sup_{t \in [0, T_{\max})} \|\nabla \sqrt{v(\cdot, t)}\|_{L^2(\mathcal{O})}^2 \leq C(u_0, v_0). \quad (3.19)$$

Theorem 3.15 (Global existence). *Let $d \in \{1, 2, 3\}$ and let u_0, v_0 satisfy the regularity assumptions of Section 3.1. Then the local solution of Theorem 3.9 is global, i.e., $T_{\max} = \infty$; for $d = 2$ it is a global classical solution, and for $d = 3$ a global weak solution in the sense of Definition 3.1.*

Proof. For $d = 1$ this is standard one-dimensional parabolic theory. For $d = 2$ and $d = 3$ this is exactly [Win12, Theorem 1.1], applied with the notation of Remark 3.2. $\square$

Lemma 3.16 (Higher integrability of u , eventual). *Let $d = 3$ and let (u, v) be the global weak solution of Theorem 3.15. For every $p \in (1, \infty)$ there exist $t_1 = t_1(p) > 0$ and $C = C(p) > 0$ such that*

$$\int_{\mathcal{O}} u^p(\cdot, t) \leq C \quad \text{for all } t > t_1. \quad (3.20)$$

Proof. See [TW12, Lemma 6.2], obtained via a weighted testing procedure with test function $\varphi(v_\varepsilon) = (2\delta - v_\varepsilon)^{-q}$ together with a Gagliardo–Nirenberg interpolation, building on the eventual smallness of v_ε . $\square$

Lemma 3.17 (L^∞ -bound for u , eventual). *Let $d = 3$ and let (u, v) be as above. There exist $T > 0$ and $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\mathcal{O})} \leq C \quad \text{for all } t \geq T. \quad (3.21)$$

Proof. See [TW12, Lemma 6.3], obtained from Lemma 3.16 via a bootstrap argument based on variation-of-constants representations and parabolic Schauder estimates for fractional powers of $-\Delta + 1$. $\square$

3.4 Why Stochastic Analysis is Different

Several of the arguments used in this chapter carry over to the stochastic setting essentially unchanged. The maximum principle underlying Lemma 3.5 is purely a property of the second equation of (3.1), which is not directly perturbed by noise in (4.1), and continues to hold pathwise. Likewise, the identity of Lemma 2.21 underlying the dissipation functional $\mathcal{D}$ is a deterministic, pointwise algebraic fact about the function v at each fixed time and is reused without modification in Chapter 4 (Section 4.2.5).

Other arguments break down and require genuinely new tools. First, local existence and uniqueness (Theorem 3.9) was obtained via a deterministic Banach fixed-point argument on a space of continuous functions. Once the equation for u is driven by a cylindrical Wiener process, the natural solution spaces become spaces of stochastic processes, and the relevant fixed-point argument can no longer be based on compactness or contraction of the solution operator itself, only of the associated family of laws, forcing the use of Skorokhod-type compactness methods in place of a direct Banach or Schauder argument (Section 2.7). Second, the energy–dissipation inequality of Lemma 3.13 was derived by directly differentiating $W(u, v)$ in time. For the stochastic system this requires Itô’s formula rather than the chain rule (Section 2.5), which produces an additional quadratic-variation (Itô correction) term that must itself be estimated and is a genuinely new source of technical difficulty absent from the

deterministic theory. Third, since solutions are no longer classical but only mild/mollified via a Yosida approximation, several of the pointwise computations of this chapter can only be justified along an approximating sequence, with the passage to the limit itself requiring uniform, expectation-valued a priori bounds rather than the deterministic uniform-in-time bounds of Corollary 3.14. These three points (stochastic compactness in place of deterministic compactness, Itô's formula in place of the chain rule, and expectation-valued rather than pathwise a priori bounds) motivate the analytical and stochastic-analytic machinery collected in Chapter 2 and are addressed systematically in the construction of Chapter 4.

4 The Stochastic Consumption Model

A large part of this chapter is orientated towards the existence proof of [HMT22a] for the one-dimensional stochastic Keller–Segel production model, adapted to the consumption structure (4.1).

4.1 Formulation of the Problem and Main Theorem

Let $\mathcal{H}$ be a Hilbert space and let $T > 0$. Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be a filtration satisfying the usual conditions i.e.,

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, i.e., $\mathcal{F}$ contains every subset of every $\mathbb{P}$ -null set,
- (ii) for each $t \geq 0$, $\mathcal{F}_t$ contains all $(\mathcal{F}, \mathbb{P})$ -null sets and
- (iii) the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is right-continuous, i.e., $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ for all $t \in [0, T)$.

Let $\mathcal{W}$ be a cylindrical Wiener process defined over $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$ (see [DZ14, Section 4.1]). (This means $\mathcal{W}$ is defined using an orthonormal basis of $\mathcal{H}$.)

We consider the following system of equations, on the fixed finite time horizon $[0, T]$, $T > 0$,

$$\begin{cases} du = (\Delta u - \operatorname{div}(u \nabla v)) \, dt + u \circ d\mathcal{W}, & \text{in } \mathcal{O} \times (0, T] \\ \partial_t v = \Delta v - uv, & \text{in } \mathcal{O} \times (0, T] \\ \partial_\nu u = \partial_\nu v = 0, & \text{on } \partial\mathcal{O} \times (0, T] \\ (u(0), v(0)) = (u_0, v_0), & \text{in } \mathcal{O}. \end{cases} \quad (4.1)$$

Both equations are nonlinear in (u, v) .

Since we model an intrinsic noise, the stochastic integral above is interpreted in the Stratonovich sense (cf. [DW14, Section 4.5.1] and [WZ65] for the Stratonovich–Itô conversion).

Let us recall the definition of a mild solution to the system (4.1). For a Banach space E the space $C^0([0, T]; E)$ is the set of all continuous and bounded functions $w: [0, T] \rightarrow E$.

Definition 4.1. We call a pair of processes (u, v) a mild solution to the system (4.1) if $u: [0, T] \times \Omega \rightarrow L^2(\mathcal{O})$ is an $\mathbb{F}$ -progressively measurable process (cf. [DZ14, Definition 3.5]) and $v: [0, T] \times \Omega \rightarrow H^1(\mathcal{O})$ is an $\mathbb{F}$ -adapted process such that $\mathbb{P}$ -a.s. $(u, v) \in C^0([0, T]; L^2(\mathcal{O})) \times$

$C^0([0, T]; H^1(\mathcal{O}))$ and satisfy for all $t \in [0, T]$ and $\mathbb{P}$ -a.s. the integral equation

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \operatorname{div}(u(s) \nabla v(s)) \, ds + \int_0^t e^{(t-s)\Delta} u(s) \circ d\mathcal{W}(s), \quad (4.2)$$

$$v(t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} u(s) v(s) \, ds. \quad (4.3)$$

Since we use compactness methods (Skorokhod and Jakubowski representation, see [BG99]) to show the existence of the solution, the given probability space will be lost, and we have to construct another probability space. In this way, we will obtain a martingale solution, which is also known as a weak solution in the probabilistic sense.

Definition 4.2. A martingale solution to the problem (4.1) is a tuple $(\mathfrak{A}, \mathcal{W}, (u, v))$ such that

- (i) $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ satisfying the usual conditions,
- (ii) $\mathcal{W}$ is a cylindrical Wiener process over $\mathfrak{A}$ on some given Hilbert space $\mathcal{H}$,
- (iii) $u: [0, T] \times \Omega \rightarrow L^2(\mathcal{O})$ is an $\mathbb{F}$ -progressively measurable process and $v: [0, T] \times \Omega \rightarrow H^1(\mathcal{O})$ is an $\mathbb{F}$ -adapted process such that (u, v) is a strong solution to (4.1) over $\mathfrak{A}$ in the sense of (4.2).

Let us consider the complete orthonormal system of the underlying Lebesgue space $L^2(\mathcal{O})$ given by the trigonometric functions (see [HMT22a]), denoted here $(e_k)_{k \in \mathbb{Z}}$ – unlike the ψ_k notation of that reference, since ψ is reserved in this thesis for the cut-off function used in the truncation of Section 4.2.2:

$$e_k(x) = \begin{cases} \sqrt{2} \sin(2\pi kx), & \text{if } k \geq 1, \, x \in \mathcal{O}, \\ 1, & \text{if } k = 0, \, x \in \mathcal{O}, \\ \sqrt{2} \cos(2\pi kx), & \text{if } k \leq -1, \, x \in \mathcal{O}. \end{cases}$$

Since $\mathcal{O} \subset \mathbb{R}$ is an interval, $(e_k)_{k \in \mathbb{Z}}$ is precisely the standard real (periodic) Fourier basis of $L^2(\mathcal{O})$; we write $\lambda_k := (2\pi k)^2$ for the associated frequencies, used below purely as a decay-rate index (via $\eta_k \sim (1 + \lambda_k)^{-\delta_1/2}$) and not as Neumann-Laplacian eigenvalues. This basis is chosen for the pointwise identity $\sum_{k \in \mathbb{Z}} (e_k^{(\delta_1)}(x))^2 = \gamma_1$ used in Section 4.2.1 below, which relies on the paired sine/cosine structure; it is *not* the same basis as the genuine Neumann-Laplacian eigenbasis $\{e_k\}_{k \in \mathbb{N}_0}$ used, e.g., as the Galerkin basis in the proof of Lemma 4.12 below – the two are unrelated orthonormal systems of $L^2(\mathcal{O})$, reused for different purposes, with $\mathbb{N}_0$ - versus $\mathbb{Z}$ -indexing serving as a visual reminder of which is meant. For the proof of the existence of the solution, the driving noise $\mathcal{W}$ needs to satisfy the regularity assumption given in the next hypothesis, ensuring that $\mathcal{W}$ takes values (in the Hilbert–Schmidt sense of Section 2.2) in a space smoother than $L^2(\mathcal{O})$.

Assumption 4.3. For a sequence $(\eta_k)_{k \in \mathbb{Z}} \subset (0, \infty)$ with $\sum_{k \in \mathbb{Z}} \eta_k^2 < \infty$ and $\eta_{-k} = \eta_k$ for all $k \in \mathbb{Z}$ (this symmetry is what makes the correction term in Section 4.2.1 below a multiple

of the identity, cf. the pointwise identity there), let $\mathcal{H}$ be the *Bessel potential space* $\mathcal{H} := \left\{ \sum_{k \in \mathbb{Z}} c_k e_k : \sum_{k \in \mathbb{Z}} \eta_k^{-2} c_k^2 < \infty \right\}$, a separable Hilbert space with orthonormal basis $e_k^{(\delta_1)} := \eta_k e_k$, $k \in \mathbb{Z}$ (the parameter δ_1 in this notation refers to the order of the potential space in the sense of $\eta_k \sim (1 + \lambda_k)^{-\delta_1/2}$, i.e. $\mathcal{H}$ interpolates between $L^2(\mathcal{O})$ and smoother Sobolev spaces depending on the decay rate of η_k). Then the embedding $\iota: \mathcal{H} \rightarrow L^2(\mathcal{O})$, $e_k^{(\delta_1)} \mapsto \eta_k e_k$, is a Hilbert–Schmidt operator in the sense of Section 2.2, since $\|\iota\|_{\text{HS}}^2 = \sum_{k \in \mathbb{Z}} \eta_k^2 < \infty$ by assumption. The Wiener process $\mathcal{W}$ is a cylindrical process on $\mathcal{H}$, i.e. $\mathcal{W} = \sum_{k \in \mathbb{Z}} \beta_k e_k^{(\delta_1)} = \sum_{k \in \mathbb{Z}} \eta_k \beta_k e_k$ for independent real Wiener processes $(\beta_k)_{k \in \mathbb{Z}}$, in the notation of Section 2.2.

Remark 4.4 (Example). As a concrete instance of Assumption 4.3, following the construction of [HMT22a, Remark 2.4], one may take $\eta_k := (1 + (2\pi k)^2)^{-\delta_1/2} = (1 + \lambda_k)^{-\delta_1/2}$ for a fixed $\delta_1 > \frac{1}{2}$, so that $\mathcal{H} = H^{\delta_1}(\mathcal{O})$ (a genuine Sobolev space, since $\delta_1 > 0$) with orthonormal basis $e_k^{(\delta_1)} = (1 + \lambda_k)^{-\delta_1/2} e_k$. The threshold $\delta_1 > \frac{1}{2}$ is exactly what is needed for $\gamma_1 = \sum_{k \in \mathbb{Z}} \eta_k^2 = \sum_{k \in \mathbb{Z}} (1 + (2\pi k)^2)^{-\delta_1} < \infty$, since $\sum_{k \in \mathbb{Z}} (1 + k^2)^{-\delta_1} < \infty$ if and only if $\delta_1 > \frac{1}{2}$; this is the source of the threshold used throughout Section 4.2 without further comment. (In [HMT22a], the same construction is used with the stronger requirement $\delta_1 > \frac{3}{2}$, since there $\mathcal{H}$ is additionally required to embed Hilbert–Schmidt into $H^\gamma(\mathcal{O})$ for some $\gamma > \frac{1}{2}$, rather than merely into $L^2(\mathcal{O})$ as in Assumption 4.3 above.)

Assumption 4.5 (Setting). Let $u_0 \in H^\alpha(\mathcal{O})$ and $v_0 \in H^1(\mathcal{O})$ be two random variables over $\mathfrak{A}$ such that

- (a) $u_0 \geq 0$ and $v_0 \geq 0$;
- (b) (u_0, v_0) is $\mathcal{F}_0$ -measurable;
- (c) $\mathbb{E}[\|u_0\|_{H^\alpha}^{2+\varepsilon_u}] < \infty$ and $\mathbb{E}[\|\nabla v_0\|_{L^2}^2] < \infty$, where $\varepsilon_u > 0$ and $\alpha \in (0, 1)$ are fixed exponents with $H^\alpha(\mathcal{O}) \xhookrightarrow{c} L^2(\mathcal{O})$ (in particular $u_0 \in L^2(\mathcal{O})$ automatically), matching the compactness argument of Lemma 4.26 (specifically the definition of M_ε and L_ε there), where this integrability is actually used;
- (d) $\sqrt{v_0} \in H^1(\mathcal{O})$ $\mathbb{P}$ -a.s. (used in the proof of Lemma 4.69 below);
- (e) there exists a deterministic $K > 0$ with $\|v_0\|_{L^\infty(\mathcal{O})} \leq K$ $\mathbb{P}$ -a.s. (used in Corollary 4.67 below, via the Hopf maximum principle bound $\|\bar{v}_n(t)\|_{L^\infty} \leq \|v_0\|_{L^\infty}$).

We now state the main result of this thesis.

Theorem 4.6 (Main result). *Let Assumption 4.3 be satisfied. Then for all initial conditions $(u_0, v_0) \in L^2(\mathcal{O}) \times H^1(\mathcal{O})$ satisfying Assumption 4.5 and for all $T > 0$, there exists a martingale solution $(\mathfrak{A}, \mathcal{W}, (u, v))$ to the system (4.1). Moreover, there exists a constant $C = C(T, u_0, v_0) > 0$ such that*

$$\mathbb{E} \left[\sup_{s \in [0, T]} \|u(s)\|_{L \log L_{\mathcal{O}}} \right] + \mathbb{E} \left[\sup_{s \in [0, T]} \|\nabla v(s)\|_{L^2_{\mathcal{O}}}^2 \right] + \mathbb{E} \left[\int_0^T \|\nabla v(s)\|_{H^1_{\mathcal{O}}}^2 ds \right] \leq C. \quad (4.4)$$

The proof of this main result is presented in the following sections.

4.2 Proof of the Main Theorem

4.2.1 Conversion to Itô Form

First of all we transform the system to the Itô Form.

The noise term $u \circ dW$ in (4.1) acts on u through the (pointwise multiplication) operator $G(u): \mathcal{H} \rightarrow L^2(\mathcal{O})$, $G(u)h := uh$, i.e. $u \circ dW = G(u) \circ dW$, and by definition of the cylindrical Wiener process (Section 2.2), $G(u) \circ dW = \sum_{k \in \mathbb{Z}} u e_k^{(\delta_1)} \circ d\beta_k$. For a scalar Stratonovich SDE $dX = b(X) dt + \sigma(X) \circ d\beta$, the equivalent Itô SDE is $dX = (b(X) + \frac{1}{2} \sigma(X) \sigma'(X)) dt + \sigma(X) d\beta$ (Section 2.6); applied term-by-term to each channel k of $G(u) \circ dW = \sum_k (u e_k^{(\delta_1)}) \circ d\beta_k$, with $\sigma_k(u) := u e_k^{(\delta_1)}$ so that $\sigma'_k(u) = e_k^{(\delta_1)}$, the correction term for channel k is $\frac{1}{2} \sigma_k(u) \sigma'_k(u) = \frac{1}{2} u (e_k^{(\delta_1)})^2$. Summing over all channels and using $\sum_{k \in \mathbb{Z}} (e_k^{(\delta_1)}(x))^2 = \gamma_1$ for a.e. $x \in \mathcal{O}$ (a pointwise identity for the trigonometric-type basis $(e_k^{(\delta_1)})_{k \in \mathbb{Z}}$ of Section 4.1, with $\gamma_1 = \sum_{k \in \mathbb{Z}} \eta_k^2 < \infty$ from Assumption 4.3), the total Itô correction drift is

$$\frac{1}{2} \sum_{k \in \mathbb{Z}} u (e_k^{(\delta_1)})^2 = \frac{\gamma_1}{2} u =: \gamma u, \quad \text{where we set } \gamma := \frac{\gamma_1}{2}.$$

Consequently, (4.1) is equivalent to the following system in Itô form, understood in the mild sense of Definition 2.32(iii):

$$\begin{cases} du = (\Delta u + \gamma u - \operatorname{div}(u \nabla v)) dt + u dW(t) \\ \partial_t v = \Delta v - uv \end{cases} \quad (4.5)$$

The remainder of this chapter works exclusively with (4.5); in particular, the operator $B := \Delta + \gamma I$ appearing throughout Section 4.2.2 onward is precisely the generator obtained by absorbing the Itô correction into the linear part of the drift.

4.2.2 Step I: The Truncated System

$$\psi(x) := \begin{cases} 0, & |x| \geq 2, \\ \in [0, 1], & |x| \in (1, 2), \\ 1, & |x| \leq 1, \end{cases}$$

and for $n \in \mathbb{N}$ let

$$\psi_n(x) := \psi\left(\frac{x}{n}\right) = \begin{cases} 0, & |x| \geq 2n, \\ \in [0, 1], & |x| \in (n, 2n), \\ 1, & |x| \leq n. \end{cases}$$

Obviously $\psi_n \rightarrow 1$ pointwise for $n \rightarrow \infty$ holds.

Definition 4.7 (Truncated system). For fixed $n \in \mathbb{N}$, we define

$$(\star) \begin{cases} du_n = (\Delta u_n + \gamma u_n - \psi_n(h^1(u_n, \cdot))\psi_n(h^2(v_n, \cdot))\psi_n(h^3(v_n, \cdot)) \operatorname{div}(u_n \nabla v_n)) dt + u_n dW(t) \\ \partial_t v_n = \Delta v_n - u_n v_n \end{cases} \quad (4.6)$$

with $\partial_\nu u_n = \partial_\nu v_n = 0$ on $\partial\mathcal{O} \times [0, T]$ and $(u_n(\cdot, 0), v_n(\cdot, 0)) = (u_0, v_0)$ in $\mathcal{O}$ which fulfill Assumption 4.5 and

$$\begin{aligned} h^1(u_n, t) &= \sup_{s \in [0, t]} \|u_n(s)\|_{L^1(\mathcal{O})} = \|u_n\|_{L^\infty(0, t; L^1_{\mathcal{O}})}, \\ h^2(v_n, t) &= \sup_{s \in [0, t]} \|\nabla v_n(s)\|_{L^2(\mathcal{O})} = \|\nabla v_n\|_{L^\infty(0, t; L^2_{\mathcal{O}})}, \\ h^3(v_n, t) &= \int_0^t \|\nabla v_n(s)\|_{H^1(\mathcal{O})}^2 ds = \|\nabla v_n\|_{L^2(0, t; H^1_{\mathcal{O}})}^2. \end{aligned}$$

Since the nonlinear terms are hard to deal with, we introduce a system depending on some $\xi \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$.

Definition 4.8 (ξ -Truncated system). For fixed $n \in \mathbb{N}$ and $\xi \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$, we define in $\mathcal{O} \times [0, T]$

$$(\star_\xi) \begin{cases} du_n = (\Delta u_n + \gamma u_n - \Psi_n^1 \Psi_n^2 \Psi_n^3 \operatorname{div}(\xi \nabla v)) dt + u_n dW(t) \\ \partial_t v = \Delta v - \xi v \end{cases} \quad (4.7)$$

with $\partial_\nu u_n = \partial_\nu v = 0$ on $\partial\mathcal{O} \times [0, T]$ and $(u_n(\cdot, 0), v(\cdot, 0)) = (u_0, v_0)$ in $\mathcal{O}$ which fulfill Assumption 4.5. Here Ψ_n^1 , Ψ_n^2 and Ψ_n^3 are defined by

$$\begin{aligned} \Psi_n^1(t) &= \psi_n(h^1(\xi, t)) = \psi_n \left(\sup_{s \in [0, t]} \|\xi(s)\|_{L^1(\mathcal{O})} \right) = \psi \left(\frac{1}{n} \|\xi\|_{L^\infty(0, t; L^1(\mathcal{O}))} \right), \\ \Psi_n^2(t) &= \psi_n(h^2(v, t)) = \psi_n \left(\sup_{s \in [0, t]} \|\nabla v(s)\|_{L^2(\mathcal{O})} \right) = \psi \left(\frac{1}{n} \|\nabla v\|_{L^\infty(0, t; L^2(\mathcal{O}))} \right), \\ \Psi_n^3(t) &= \psi_n(h^3(v, t)) = \psi_n \left(\int_0^t \|\nabla v(s)\|_{H^1(\mathcal{O})}^2 ds \right) = \psi \left(\frac{1}{n} \|\nabla v\|_{L^2(0, t; H^1(\mathcal{O}))}^2 \right). \end{aligned}$$

Note that v is independent from n .

Lemma 4.9. For these three functions, $a \geq b > 0$ and $\alpha \in (0, 2]$, $q \in (0, \frac{2}{\alpha})$ the following are true for all $t \in [0, T]$

$$\begin{aligned} (i) \quad & \Psi_n^1(t)^a \sup_{s \in [0, t]} \|\xi(s)\|_{L^1}^b \leq (2n)^b, \\ (ii) \quad & \Psi_n^1(t)^a \sup_{s \in [0, t]} \|\nabla v(s)\|_{L^2}^b \leq (2n)^b, \\ (iii) \quad & \int_0^t \Psi_n^3(s) \|\nabla v(s)\|_{H^1}^2 ds \leq 2n, \\ (iv) \quad & \int_0^t \Psi_n^3(s)^q \|\nabla v(s)\|_{H^1}^{q\alpha} ds \leq \max(1, T) (2n)^{\frac{q\alpha}{2}}. \end{aligned}$$

Proof. For all $t \in [0, T]$ it holds that

$$\begin{aligned} \Psi_n^1(t)^a \sup_{s \in [0, t]} \|\xi(s)\|_{L^1}^b &\leq 1^{a-b} \Psi_n^1(t)^b \sup_{s \in [0, t]} \|\xi(s)\|_{L^1}^b \leq \left(\Psi_n^1(t) \sup_{s \in [0, t]} \|\xi(s)\|_{L^1} \right)^b \\ &\leq \left(\mathbf{1}_{\{\sup_{s \in [0, t]} \|\xi(s)\|_{L^1} \leq 2n\}} \sup_{s \in [0, t]} \|\xi(s)\|_{L^1} \right)^b \leq (2n)^b. \end{aligned}$$

The same calculation leads to (ii). Let $\tau = \inf \{s > 0 : \int_0^s \|\nabla v(\sigma)\|_{H^1}^2 d\sigma \geq 2n\}$, then

$$\begin{aligned} \int_0^t \Psi_n^3(s) \|\nabla v(s)\|_{H^1}^2 ds &\leq \int_0^t \mathbf{1}_{\{\int_0^s \|\nabla v(\sigma)\|_{H^1}^2 d\sigma \leq 2n\}} \|\nabla v(s)\|_{H^1}^2 ds \\ &\leq \int_0^{t \wedge \tau} \|\nabla v(s)\|_{H^1}^2 ds \leq \int_0^\tau \|\nabla v(s)\|_{H^1}^2 ds \leq 2n. \end{aligned}$$

Using Hölder with $p = \frac{2}{q\alpha}$ yields

$$\begin{aligned} \int_0^t \Psi_n^3(s)^q \|\nabla v(s)\|_{H^1}^{q\alpha} ds &\leq t^{\frac{1}{p'}} \left(\int_0^t \Psi_n^3(s)^{\frac{2}{\alpha}} \|\nabla v(s)\|_{H^1}^2 ds \right)^{\frac{q\alpha}{2}} \\ &\leq T^{1-\frac{q\alpha}{2}} \left(\int_0^t \Psi_n^3(s) \|\nabla v(s)\|_{H^1}^2 ds \right)^{\frac{q\alpha}{2}} \\ &\leq \max(1, T) (2n)^{\frac{q\alpha}{2}}. \end{aligned} \quad \square$$

Lemma 4.10. For every $q \in [1, 2]$, $\xi \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ and $t \in [0, T]$, $\mathbb{E} \left[\sup_{s \in [0, t]} \|\xi(s)\|_{L^2_{\mathcal{O}}}^q \right]$ is bounded and hence $\sup_{s \in [0, t]} \|\xi(s)\|_{L^2_{\mathcal{O}}}^q < \infty$ $\mathbb{P}_n$ -a.s.

Proof. For $q = 2$ the statement is obvious. For $q < 2$ we can use Hölder in the probability space and obtain the statement. $\square$

We next want to prove existence and uniqueness of this system. For this we first obtain some estimates for any v fulfilling the second equation of the system.

Lemma 4.11. In $d = 1$, for every v fulfilling the second equation of (4.7), there exists a constant $C = C(T) > 0$

$$\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \int_0^t \|\Delta v(s)\|_{L^2}^2 ds \leq C \|v_0\|_{H^1_{\mathcal{O}}}^2 \quad \mathbb{P}_n\text{-a.s. for all } t \in [0, T].$$

Proof. Using Gagliardo-Nirenberg in $d \in \{1, 2\}$, Young, we can calculate

$$\begin{aligned} \frac{1}{2} \partial_t \|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 &= \int_{\mathcal{O}} \xi(t) v(t)^2 \leq \|\xi(t)\|_{L^2} \|v(t)\|_{L^4}^2 \\ &\leq \|\xi(t)\|_{L^2} \|\nabla v(t)\|_{L^2}^{\frac{d}{2}} \|v(t)\|_{L^2}^{2(1-\frac{d}{4})} \leq \frac{d}{4} \|\nabla v(t)\|_{L^2}^2 + \frac{4-d}{4} \|\xi(t)\|_{L^2}^{\frac{4}{4-d}} \|v(t)\|_{L^2}^2, \end{aligned}$$

which with Gronwall and $\sup_{s \in [0, t]} \|\xi(s)\|_{L^2_{\mathcal{O}}}^{\frac{4}{4-d}} < \infty$ $\mathbb{P}_n$ -a.s. (since $\frac{4}{4-d} \leq 2$) gives us

$$\|v(t)\|_{L^2}^2 \leq e^{\frac{4-d}{2} \int_0^t \|\xi(s)\|_{L^2}^{\frac{4}{4-d}} ds} \|v_0\|_{L^2}^2 \leq C_1(T, d) \|v_0\|_{L^2}^2 \quad \mathbb{P}_n\text{-a.s.}$$

For the second inequality we also just calculate with Young, Hölder, Sobolev embedding $H^1 \hookrightarrow L^\infty$ in $d = 1$

$$\begin{aligned} \frac{1}{2} \partial_t \|\nabla v(t)\|_{L^2}^2 + \|\Delta v(t)\|_{L^2}^2 &= - \int_{\mathcal{O}} \partial_t v(t) \Delta v(t) + \|\Delta v(t)\|_{L^2}^2 = \int_{\mathcal{O}} \xi(t) v(t) \Delta v(t) \\ &\leq \frac{1}{2} \|\Delta v(t)\|_{L^2}^2 + \frac{1}{2} \|\xi(t) v(t)\|_{L^2}^2 \leq \frac{1}{2} \|\Delta v(t)\|_{L^2}^2 + \frac{1}{2} \|\xi(t)\|_{L^2}^2 \|v(t)\|_{L^\infty}^2 \\ &\leq \frac{1}{2} \|\Delta v(t)\|_{L^2}^2 + \frac{1}{2} \|\xi(t)\|_{L^2}^2 \|v(t)\|_{H^1}^2 = \frac{1}{2} \|\Delta v(t)\|_{L^2}^2 + \frac{1}{2} \|\xi(t)\|_{L^2}^2 (\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2) \\ &\leq \frac{1}{2} \|\Delta v(t)\|_{L^2}^2 + \frac{C_1}{2} \|\xi(t)\|_{L^2}^2 \|v_0\|_{L^2}^2 + \frac{1}{2} \|\xi(t)\|_{L^2}^2 \|\nabla v(t)\|_{L^2}^2, \end{aligned}$$

which leads us to

$$\begin{aligned} &\partial_t \left(\|\nabla v(t)\|_{L^2}^2 + \int_0^t \|\Delta v(s)\|_{L^2}^2 ds \right) \\ &\leq C_1 \|\xi(t)\|_{L^2}^2 \|v_0\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2 \left(\|\nabla v(t)\|_{L^2}^2 + \int_0^t \|\Delta v(s)\|_{L^2}^2 ds \right), \end{aligned}$$

which again with Gronwall gives us

$$\begin{aligned} &\|\nabla v(t)\|_{L^2}^2 + \int_0^t \|\Delta v(s)\|_{L^2}^2 ds \\ &\leq e^{\int_0^t \|\xi(s)\|_{L^2}^2 ds} \left(\|\nabla v_0\|_{L^2}^2 + C_1 \|v_0\|_{L^2}^2 \int_0^t \|\xi(s)\|_{L^2}^2 e^{-\int_0^s \|\xi(\sigma)\|_{L^2}^2 d\sigma} ds \right) \\ &\leq C_2(T) \left(\|\nabla v_0\|_{L^2}^2 + C_1 C_3(T) \|v_0\|_{L^2}^2 \right) \quad \mathbb{P}_n\text{-a.s.}, \end{aligned}$$

which proves the claim. $\square$

Lemma 4.12. *There exists a solution for the second component in system (4.7) fulfilling*

$$v \in L^\infty(0, T; H^1(\mathcal{O})) \cap L^2(0, T; H^2(\mathcal{O})) \quad \mathbb{P}\text{-a.s.}$$

Proof. We will work pathwise, i.e., for fixed $\omega \in \Omega$ and the proof will be split into two parts.

1. Galerkin-Equations: Let $\{e_k\}_{k \in \mathbb{N}_0}$ be an orthonormal basis of $L^2(\mathcal{O})$ existing of eigenfunctions of $-\Delta$ with Neumann boundary conditions. Let us define the orthogonal projection

$$P_m: L^2(\mathcal{O}) \rightarrow V_m, \quad w \mapsto \sum_{k=0}^m \langle w, e_k \rangle_{L^2_{\mathcal{O}}} e_k, \quad \text{with } V_m = \text{span}\{e_0, \dots, e_m\}.$$

If we now set $v_m(t) = \sum_{k=0}^m c_k(t) e_k$ we can see that

$$\partial_t v_m = P_m(\Delta v_m - \xi v_m), \quad v_m(0) = P_m v_0,$$

is equivalent to

$$\partial_t c_k(t) = -\lambda_k c_k(t) - \sum_{j=0}^m A_{kj}(t) c_j(t), \quad c_k(0) = \langle v_0, e_k \rangle_{L^2_{\mathcal{O}}}, \quad k \in \{0, \dots, m\},$$

with $A_{kj}(t) = \langle \xi(t) e_k, e_j \rangle_{L^2_{\mathcal{O}}}$ is a linear ODE-system in $(c_0, \dots, c_m)$. Since $|A_{kj}(t)| \leq C_1 \|\xi\|_{L^2_{\mathcal{O}}}$, we have $A_{kj} \in L^\infty(0, T)$, hence $A_{kj} \in L^1(0, T)$ and henceforth by the Carathéodory existence theorem (see [Har82, Chapter II, Theorem 1.1]) there exists a solution to this system for every m .

2. Limiting process: First, since $\sum_{k=0}^{\infty} |\langle v_0, e_k \rangle_{L^2_{\mathcal{O}}}|^2 = \|v_0\|_{L^2_{\mathcal{O}}}^2 < \infty$ we obtain

$$\begin{aligned} \|v_0 - v_m(0)\|_{L^2_{\mathcal{O}}}^2 &= \|v_0 - P_m v_0\|_{L^2_{\mathcal{O}}}^2 = \left\| \sum_{k=0}^{\infty} \langle v_0, e_k \rangle_{L^2_{\mathcal{O}}} e_k - \sum_{k=0}^m \langle v_0, e_k \rangle_{L^2_{\mathcal{O}}} e_k \right\| \\ &\leq \sum_{k=m+1}^{\infty} |\langle v_0, e_k \rangle_{L^2_{\mathcal{O}}}|^2 \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

The estimate in Lemma 4.11 holds for v_m because $\|P_m\|_{\text{op}} \leq 1$. Hence $\{v_m\}$ is bounded in $L_T^2 H_{\mathcal{O}}^2$ and $\{\Delta v_m\}$ is bounded in $L_T^2 L_{\mathcal{O}}^2$. Since with the Sobolev embedding $H^1 \hookrightarrow L^\infty$ in $d = 1$

$$\|\xi v_m\|_{L_T^2 L_{\mathcal{O}}^2}^2 \leq \|\xi\|_{L_{\mathcal{O}}^2} \|v_m\|_{L_{\mathcal{O}}^\infty}^2 \leq C_4 \|\xi\|_{L_{\mathcal{O}}^2} \|v_m\|_{H_{\mathcal{O}}^1}^2 \leq C_4 T \|\xi\|_{L_T^\infty L_{\mathcal{O}}^2}^2 \|v_m\|_{L_T^\infty H_{\mathcal{O}}^1}^2,$$

also $\{\xi v_m\}$ is bounded in $L_T^2 L_{\mathcal{O}}^2$ and hence also $\{\partial_t v_m\}$. With an Aubin–Lions–Simon compactness argument ([Sim87]) from $H_{\mathcal{O}}^2 \hookrightarrow H_{\mathcal{O}}^1 \hookrightarrow L_{\mathcal{O}}^2$, $\{v_m\}$ bdd. in $L_T^2 H_{\mathcal{O}}^2$ and $\{\partial_t v_m\}$ bdd. in $L_T^2 L_{\mathcal{O}}^2$ we can conclude that $\{v_m\}$ is precompact in $L_T^2 H_{\mathcal{O}}^1$ and hence there exists a subsequence v_{m_i} (we just keep writing v_m) with $v_m \rightarrow \tilde{v}$ strongly in $L_T^2 H_{\mathcal{O}}^1$ for some $\tilde{v}$. For $\phi \in L^2(\mathcal{O})$ und $\varphi \in C_c^\infty(0, T)$ we have

$$\begin{aligned} \left| \int_0^T \langle \partial_t(v_m - \tilde{v})(t), \phi \rangle_{L_{\mathcal{O}}^2} \varphi(t) dt \right| &= \left| \int_0^T \langle v_m(t) - \tilde{v}(t), \phi \rangle_{L_{\mathcal{O}}^2} \varphi'(t) dt \right| \\ &\leq \|v_m - \tilde{v}\|_{L_T^2 L_{\mathcal{O}}^2} \|\phi\|_{L_{\mathcal{O}}^2} \|\varphi'\|_{L_T^2} \rightarrow 0, \end{aligned}$$

for $m \rightarrow \infty$ and so we can put together

$$\int_0^T \langle \partial_t v_m + \xi v_m, \Phi \rangle_{L_{\mathcal{O}}^2} + \int_0^T \langle \nabla v_m, \nabla \Phi \rangle_{L_{\mathcal{O}}^2} \rightarrow \int_0^T \langle \partial_t \tilde{v} + \xi \tilde{v}, \Phi \rangle_{L_{\mathcal{O}}^2} + \int_0^T \langle \nabla \tilde{v}, \nabla \Phi \rangle_{L_{\mathcal{O}}^2},$$

for $m \rightarrow \infty$ for any $\Phi \in L_T^2 H_{\mathcal{O}}^1$. Hence $\tilde{v}$ is a weak solution to $\partial_t v = \Delta v - \xi v$, $v(0) = v_0$. The regularity statement follows directly from Lemma 4.11. $\square$

Lemma 4.13. *The solution from Lemma 4.12 is $\mathbb{P}_n$ -a.s. unique.*

Proof. Assume there are two different solutions v_1, v_2 and set $w = v_1 - v_2$ which then fulfills $\partial_t w = \Delta w - \xi w$, $\partial_\nu w|_{\partial \mathcal{O}} = 0$, $w(0) = 0$. The same calculation as in Lemma 4.11 leads to

$$\|w(t)\|_{H_{\mathcal{O}}^1}^2 \leq C \|w(0)\|_{H_{\mathcal{O}}^1}^2 = 0 \quad \mathbb{P}_n\text{-a.s. for all } t \in [0, T],$$

hence $v_1 = v_2$ $\mathbb{P}_n$ -a.s. in $L_T^\infty H_{\mathcal{O}}^1$. $\square$

Lemma 4.14. *For fixed $n \in \mathbb{N}$ and v given by Lemma 4.12 and Lemma 4.13, the first equation of (4.7) has a unique solution u_n .*

Proof. With ξ and v fixed (the latter by Lemma 4.12, independently of u_n , since the second equation of (4.7) does not involve u_n), the first equation of (4.7) reads

$$du_n = ((\Delta + \gamma I)u_n + F(t)) dt + u_n d\mathcal{W}(t), \quad F(t) := -\Psi_n^1(t)\Psi_n^2(t)\Psi_n^3(t) \operatorname{div}(\xi(t)\nabla v(t)),$$

which is linear in u_n : the drift operator $A = \Delta + \gamma I$ generates an analytic semigroup (Lemma 2.19), the noise coefficient $G(u_n)h := u_nh$ is bounded and linear (as in Section 4.2.1), and the forcing F does not depend on u_n . Moreover F has locally integrable trajectories, indeed $F \in L^2(0, T; L^2(\mathcal{O}))$ $\mathbb{P}_n$ -a.s.: by the same estimate as for I_2 in the proof of Lemma 4.17, $\|F(t)\|_{L^2}$ is controlled using the cutoffs $\Psi_n^1, \Psi_n^2, \Psi_n^3$, which by construction keep $\|\xi\|_{L^1}, \|\nabla v\|_{L^2}$ and $\int_0^t \|\nabla v\|_{H^1}^2$ bounded by $2n$ (Lemma 4.9). Existence and uniqueness of a mild solution u_n therefore follow from Theorem 2.33, applied with $X_0 = u_0, f = F$ and noise operator $G(u_n)$, exactly as for the linear system (4.24) of Section 4.2.4. $\square$

Lemma 4.15. *For any $q \in [0, 2)$ the factor of the semigroup divergence smoothing between L^2 and L^2 (Lemma 2.19(iv)) is an element of $L^q(0, T)$, i.e., for any $\alpha > 0$*

$$r(s) = \left(1 + s^{-\frac{1}{2}}\right)e^{-\alpha s} \implies \int_0^t r(t-s)^q ds = \int_0^t r(s)^q ds < \infty \quad \text{for all } t \in [0, T].$$

Proof. For $q = 0$ the statement is obvious. For $q \in (0, 2)$ we can calculate

$$\begin{aligned} \int_0^t r(s)^q ds &= \int_0^t (1 + s^{-1/2})^q e^{-q\alpha s} ds = \int_0^{\sqrt{t}} (1 + 1/z)^q e^{-q\alpha z^2} \cdot 2z dz \\ &= \int_0^1 (1 + 1/z)^q e^{-q\alpha z^2} 2z dz + \int_1^{\sqrt{t}} (1 + 1/z)^q e^{-q\alpha z^2} \cdot 2z dz \\ &\leq 2^{q+1} \int_0^1 z^{1-q} dz + 2^q \int_1^\infty 2z e^{-q\alpha z^2} dz = \frac{2^{q+1}}{2-q} + \frac{2^q}{q\alpha} e^{-q\alpha}. \end{aligned} \quad \square$$

Self-mapping property

First we need a set which is mapped into itself by $\mathfrak{T}_n$ for each $n \in \mathbb{N}$. For this let us fix $n \in \mathbb{N}$.

Definition 4.16. Since for fixed $\xi \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ we obtained a unique solution pair (u_n, v) we can define the operators $\mathfrak{T}_n$ and $\mathfrak{R}$ which operate on $\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ and map ξ to u_n and v respectively. For $B = \Delta + \gamma I$ the variation of constant formula yields

$$\begin{aligned} (\mathfrak{T}_n \xi)(t) = u_n(t) &= e^{tB} u_0 - \int_0^t e^{(t-s)B} \Psi_n^1(s) \Psi_n^2(s) \Psi_n^3(s) \operatorname{div}(\xi(s) \nabla v(s)) ds \\ &\quad + \int_0^t e^{(t-s)B} u_n(s) d\mathcal{W}(s), \end{aligned} \quad (4.8)$$

and

$$(\mathfrak{R} \xi)(t) = v(t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \xi(s) v(s) ds. \quad (4.9)$$

Lemma 4.17 (Self-mapping property of $\mathfrak{T}_n$). *Let (u_0, v_0) fulfill Assumption 4.5. Then there exists $R = R(n, u_0) > 0$ with $\mathfrak{T}_n: \mathcal{C}_R \rightarrow \mathcal{C}_R$ where*

$$\mathcal{C}_R := \{\xi \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2) \mid \|\xi\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2 \leq R\}. \quad (4.10)$$

Proof. Let us fix $\xi \in \mathcal{C}_R \subset \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)$ and let $(u_n, v) = (\mathfrak{T}_n(\xi), \mathbb{R}(\xi))$. Now let $\xi \in \mathcal{C}_R \subset \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)$. For

$$\begin{aligned} I_1(s) &:= \|e^{sB} u_0\|_{L^2(\mathcal{O})}^2 \\ I_2(s) &:= \left\| \int_0^s e^{(s-\sigma)B} \Psi_n^1(\sigma) \Psi_n^2(\sigma) \Psi_n^3(\sigma) \operatorname{div}(\xi(\sigma) \nabla v(\sigma)) \, d\sigma \right\|_{L^2(\mathcal{O})}^2 \\ I_3(s) &:= \left\| \int_0^s e^{(s-\sigma)B} u_n(\sigma) \, d\mathcal{W}(\sigma) \right\|_{L^2(\mathcal{O})}^2 \end{aligned}$$

it follows with the Minkowski inequality (Theorem 2.7) and $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ that

$$\frac{1}{3} \mathbb{E} \left[\sup_{s \in [0, t]} \|u_n(s)\|_{L^2(\mathcal{O})}^2 \right] \leq \mathbb{E} \left[\sup_{s \in [0, t]} I_1(s) \right] + \mathbb{E} \left[\sup_{s \in [0, t]} I_2(s) \right] + \mathbb{E} \left[\sup_{s \in [0, t]} I_3(s) \right].$$

For I_1 : With Lemma 2.19(i) for $p = 2$ we have

$$I_1(s) = \|e^{s\Delta + s\gamma I} u_0\|_{L^2}^2 = e^{2\gamma s} \|e^{s\Delta} u_0\|_{L^2}^2 \leq e^{2\gamma s} \|u_0\|_{L^2}^2.$$

For I_2 : By Lemma 2.8, Lemma 2.19(vi) and the product estimate of Lemma 2.16 together with the interpolation inequalities of Lemmas 2.15 and 2.14, embedding $L^1 \hookrightarrow H^{-\varepsilon}$ for $\varepsilon \in (\frac{1}{2}, 1)$, $\delta \in (0, \frac{1}{2})$ (Lemma 2.13(vi), valid here since $d = 1$ so $\varepsilon > \frac{1}{2} = \frac{d}{2}$), we have

$$\begin{aligned} (I_2(s))^{\frac{1}{2}} &\leq \int_0^s \|e^{(s-\sigma)B} \Psi_n^1(\sigma) \Psi_n^2(\sigma) \Psi_n^3(\sigma) \operatorname{div}(\xi(\sigma) \nabla v(\sigma))\|_{L^2(\mathcal{O})} \, d\sigma \\ &= \int_0^s \Psi_n^1(\sigma) \Psi_n^2(\sigma) \Psi_n^3(\sigma) \|e^{(s-\sigma)B} \operatorname{div}(\xi(\sigma) \nabla v(\sigma))\|_{L^2(\mathcal{O})} \, d\sigma \\ &\leq \int_0^s \Psi_n^1(\sigma) \Psi_n^2(\sigma) \Psi_n^3(\sigma) r(s-\sigma) \|\xi(\sigma) \nabla v(\sigma)\|_{L^2(\mathcal{O})} \, d\sigma \\ &\leq \int_0^s \Psi_n^1(\sigma) \Psi_n^2(\sigma) \Psi_n^3(\sigma) r(s-\sigma) \|\xi(\sigma)\|_{H^{-\varepsilon}}^{\frac{\delta}{\varepsilon}} \|\xi(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \|\nabla v(\sigma)\|_{L^2}^{\frac{1}{2}-\delta} \|\nabla v(\sigma)\|_{H^1}^{\delta+\frac{1}{2}} \, d\sigma \\ &\leq \sup_{\sigma \in [0, s]} \left(\Psi_n^1(\sigma) \|\xi(\sigma)\|_{H^{-\varepsilon}}^{\frac{\delta}{\varepsilon}} \right) \cdot \sup_{\sigma \in [0, s]} \left(\Psi_n^2(\sigma) \|\nabla v(\sigma)\|_{L^2}^{\frac{1}{2}-\delta} \right) \cdot \sup_{\sigma \in [0, s]} \|\xi(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \\ &\quad \times \int_0^s r(s-\sigma) \Psi_n^3(\sigma) \|\nabla v(\sigma)\|_{H^1}^{\delta+\frac{1}{2}} \, d\sigma \\ &\leq (2n)^{\frac{\delta}{\varepsilon}} (2n)^{\frac{1}{2}-\delta} \sup_{\sigma \in [0, s]} \|\xi(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \int_0^s r(s-\sigma) \Psi_n^3(\sigma) \|\nabla v(\sigma)\|_{H^1}^{\delta+\frac{1}{2}} \, d\sigma. \end{aligned}$$

Where with Hölder ($p = \frac{4}{2\delta+1}$ and $q = \frac{4}{3-2\delta}$)

$$\begin{aligned} \int_0^s r(s-\sigma) \Psi_n^3(\sigma) \|\nabla v(\sigma)\|_{H^1}^{\delta+\frac{1}{2}} d\sigma &= \int_0^s r(s-\sigma) (\Psi_n^3(\sigma))^{\frac{1}{q}} (\Psi_n^3(\sigma))^{\frac{1}{p}} \|\nabla v(\sigma)\|_{H^1}^{\frac{2}{p}} d\sigma \\ &\leq \left(\int_0^s (r(s-\sigma))^q \Psi_n^3(\sigma) d\sigma \right)^{\frac{1}{q}} \cdot \left(\int_0^s \Psi_n^3(\sigma) \|\nabla v(\sigma)\|_{H^1}^2 d\sigma \right)^{\frac{2\delta+1}{4}} \\ &\leq C \cdot (2n)^{\frac{2\delta+1}{4}} \end{aligned}$$

where the last inequality is Lemma 4.15 ($q < 2$) and Lemma 4.9.

For I_3 we can use Theorem 2.35 for the case $p = 2$, $\rho = 0$ and obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} I_3(s) \right] &\leq C \mathbb{E} \left[\int_0^t \overbrace{\|e^{(t-s)(\Delta+\gamma I)} u_n(s)\|_{L^2}^2}^{\leq e^{2\gamma(t-s)} \|u_n(s)\|_{L^2}^2} ds \right] \\ &\leq C(T) \mathbb{E} \left[\int_0^t \|u_n(s)\|_{L^2}^2 ds \right] \leq C(T) \int_0^t \mathbb{E} \left[\sup_{\sigma \in [0, s]} \|u_n(\sigma)\|_{L^2}^2 \right] ds. \end{aligned}$$

Hence in total we have

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} \|u_n(s)\|_{L^2(\mathcal{O})}^2 \right] &\leq 3 \mathbb{E} [\|u_0\|_{L^2(\mathcal{O})}^2] + C_1(T, n) \mathbb{E} \left[\sup_{s \in [0, t]} \|\xi(s)\|_{L^2}^{2-2\frac{\delta}{\varepsilon}} \right] \\ &\quad + C_2(T) \mathbb{E} \left[\sup_{\sigma \in [0, s]} \|u_n(\sigma)\|_{L^2}^2 \right] ds \end{aligned}$$

and for $y(t) = \mathbb{E} \left[\sup_{s \in [0, t]} \|u_n(s)\|_{L^2}^2 \right]$ Gronwall's inequality (Theorem 2.17, applied pointwise in ω) yields with $\delta = \frac{\varepsilon}{2}$

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} \|u_n(s)\|_{L^2(\mathcal{O})}^2 \right] &\leq e^{C(T)t} \left(3 \mathbb{E} [\|u_0\|_{L^2(\mathcal{O})}^2] + C_2 \mathbb{E} \left[\sup_{s \in [0, t]} \|\xi(s)\|_{L^2}^{4-2} \right]^{\frac{1}{2}} \right) \\ &\leq C_4(T, n) \left(\mathbb{E} [\|u_0\|_{L^2(\mathcal{O})}^2] + \sqrt{R} \right). \end{aligned} \tag{4.11}$$

If the right side here would be smaller than R , then $\|u_n\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2_{\mathcal{O}})}^2 < R$ and hence $\mathfrak{T}_n \xi = u_n \in \mathcal{C}_R$. Since for every $n \in \mathbb{N}$ we can find such an $R = R(n, u_0) > 0$ fulfilling (4.11) $\leq R$ the proof of Lemma 4.17 is complete. $\square$

Tightness

For the authors of [HMT22a] as well as us it was not possible to directly show compactness of the operator $\mathfrak{T}_n$. This and the fact, that the probability spaces we work on might change leads us to the following steps, which are quite close to a compactness proof.

The next property we need is tightness of the family of laws of $\mathfrak{T}_n \xi$ for $\xi \in \mathcal{C}_R$ for each $n \in \mathbb{N}$. For that we first collect a number of auxiliary inequalities, which will then lead us to the tightness statement of Lemma 4.26. So let us fix $n \in \mathbb{N}$.

Definition 4.18. Let A be an operator generating a semigroup (in our case either Δ or

$B = \Delta + \gamma I$. For $\eta \in L^q(0, T; L^2(\mathcal{O}))$ and $\zeta \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ we define

$$(\mathfrak{F}_A \eta)(t) := \int_0^t e^{(t-s)A} \eta(s) \, ds, \quad \text{and} \quad (4.12)$$

$$(\mathfrak{S}_A \zeta)(t) := \int_0^t e^{(t-s)A} \zeta(s) \, d\mathcal{W}(s), \quad \text{for } t \in [0, T]. \quad (4.13)$$

Lemma 4.19. *For $\eta \in L^q(0, T; L^2(\mathcal{O}))$, $\zeta \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ and $\mathfrak{F}_B, \mathfrak{S}_B$ from Definition 4.18, the following statement holds.*

For $\rho, \alpha \in \mathbb{R}$, $\beta > 0$, $q \geq 1$ chosen such that $\beta + \frac{\alpha}{2} + \rho < 1 - \frac{1}{q}$ and $\eta \in L^q(0, T; H^{-\rho}(\mathcal{O}))$, there exists $C = C(T, \alpha, \beta, \rho, q, \mathcal{O}) > 0$ such that

$$\|\mathfrak{F}_B \eta\|_{C^\beta([0, T]; H^\alpha)} \leq C \|\eta\|_{L^q(0, T; H^{-2\rho})}. \quad (4.14)$$

Proof. First notice that

$$e^{-\gamma T} (\mathfrak{F}_B \eta)(t) \leq \int_0^t e^{(t-s)\Delta} \eta(s) \, ds =: (\mathfrak{F}_\Delta \eta)(t).$$

Looking at the difference (WLOG $s \leq t$)

$$\mathfrak{F}_\Delta \eta(t) - \mathfrak{F}_\Delta \eta(s) = \int_s^t e^{(t-\sigma)\Delta} \eta(\sigma) \, d\sigma + (e^{(t-s)\Delta} + I) \int_0^s e^{(s-\sigma)\Delta} \eta(\sigma) \, d\sigma,$$

we see for the first term that I_1 with semigroup inequalities

$$\begin{aligned} \|I_1\|_{H_\mathcal{O}^\alpha} &\leq \int_s^t \|e^{(t-\sigma)\Delta} \eta(\sigma)\| \, d\sigma \leq C_1 \int_s^t (t-\sigma)^{-\frac{\alpha+2\rho}{2}} \|\eta(\sigma)\|_{H^{-2\rho}} \, d\sigma \\ &\leq C_1 \left(\int_s^t (t-\sigma)^{-\frac{\alpha+2\rho}{2} q'} \, d\sigma \right)^{\frac{1}{q'}} \|\eta\|_{L^q(s, t; H_\mathcal{O}^{-2\rho})} \leq C_2 |t-s|^{\frac{1}{q'} - \frac{\alpha+2\rho}{2}} \|\eta\|_{L_T^q H_\mathcal{O}^{-2\rho}} \\ &\leq C_2 |t-s|^\beta \|\eta\|_{L_T^q H_\mathcal{O}^{-2\rho}}, \end{aligned}$$

since $\frac{\alpha}{2} + \rho < 1 - \frac{1}{q} = \frac{1}{q'}$. For the second term we can see that again with semigroup inequalities

$$\begin{aligned} \|(e^{(t-s)\Delta} + I) I_2\|_{H_\mathcal{O}^\alpha} &\leq C_2 |t-s|^\beta \|I_3\|_{H_\mathcal{O}^{\alpha+2\beta}} \leq C_2 |t-s|^\beta \int_0^s \|e^{(s-\sigma)\Delta} \eta(\sigma)\|_{H_\mathcal{O}^{\alpha+2\beta}} \, d\sigma \\ &\leq C_3 |t-s|^\beta \int_0^s (s-\sigma)^{-(\beta+\frac{\alpha}{2}+\rho)} \|\eta(\sigma)\|_{H_\mathcal{O}^{-2\rho}} \, d\sigma \\ &\leq C_3 |t-s|^\beta \left(\int_0^s (s-\sigma)^{-(\beta+\frac{\alpha}{2}+\rho) q'} \, d\sigma \right)^{\frac{1}{q'}} \|\eta\|_{L^q(0, s; H_\mathcal{O}^{-2\rho})} \\ &\leq C_4 |t-s|^\beta s^{\frac{1}{q'} - (\beta+\frac{\alpha}{2}+\rho)} \|\eta\|_{L_T^q H_\mathcal{O}^{-2\rho}}, \end{aligned}$$

where the term with s is bounded since the exponent is positive by assumption. This completes the proof of the statement. $\square$

Lemma 4.20. *For $\eta = \Psi_n^1 \Psi_n^2 \Psi_n^3 \operatorname{div}(\xi \nabla v)$, there exists a constant $C = C(T, n)$ such that*

$$\|\eta\|_{L^q(0, T; H^{-2\rho})} \leq C \sup_{t \in [0, T]} \|\xi(t)\|_{L^2}.$$

Proof. By the Sobolev embedding (Lemma 2.13(vi)) (for $d = 1$) $L^1(\mathcal{O}) \hookrightarrow H^{-(2\rho-1)}(\mathcal{O})$ for $\rho > \frac{3}{4}$ we obtain

$$\|\operatorname{div}(\xi(s)\nabla v(s))\|_{H^{-2\rho}} \leq \|\xi(s)\nabla v(s)\|_{H^{-2\rho+1}} \leq \|\xi(s)\nabla v(s)\|_{L^1} \leq \|\xi(s)\|_{L^2} \|\nabla v(s)\|_{L^2}$$

(The first inequality holds since $\operatorname{div}: H^s(\mathcal{O})^d \rightarrow H^{s-1}(\mathcal{O})$ is bounded with norm 1 for every $s \in \mathbb{R}$: differentiation lowers the order of a Sobolev space by exactly one, directly from the definition of the H^s -norm via the spectral/Fourier characterization applied componentwise.) Let us put $\eta(s) = \Psi_n^1(s)\Psi_n^2(s)\Psi_n^3(s)\operatorname{div}(\xi(s)\nabla v(s))$, $s \in [0, T]$. For $q \geq 2$ we can see that

$$\begin{aligned} \|\eta\|_{L^q(0,T;H^{-2\rho})}^q &= \int_0^T \|\eta(s)\|_{H^{-2\rho}}^q ds = \int_0^T (\Psi_n^1(s)\Psi_n^2(s)\Psi_n^3(s)\|\operatorname{div}(\xi(s)\nabla v(s))\|_{H^{-2\rho}})^q ds \\ &\leq \left(\sup_{s \in [0,T]} \Psi_n^1(s)\Psi_n^3(s) \cdot \sup_{s \in [0,T]} \Psi_n^2(s)\|\nabla v(s)\|_{L^2} \cdot \sup_{s \in [0,T]} \|\xi(s)\|_{L^2} \right)^q \int_0^T 1 ds \\ &\leq T \left(1 \cdot 2n \cdot \sup_{s \in [0,T]} \|\xi(s)\|_{L^2} \right)^q. \end{aligned} \quad \square$$

Lemma 4.21. *There exists $C = C(T) > 0$ such that for all $w \in L^2(\mathcal{O})$ and $s, t \in [0, T]$,*

$$\|e^{t(\Delta+\gamma I)}w - e^{s(\Delta+\gamma I)}w\|_{L^2} \leq C|t-s| \cdot \|w\|_{L^2},$$

i.e. the mapping $t \mapsto e^{t(\Delta+\gamma I)}w$ is Lipschitz (in particular Hölder-continuous with exponent 1) in t .

Proof. By Lemma 2.19, the mean value theorem and $1 - e^{-x} \leq x$, we obtain

$$\begin{aligned} \|e^{t(\Delta+\gamma I)}w - e^{s(\Delta+\gamma I)}w\|_{L^2} &\leq e^{\gamma t} \|e^{t\Delta}w - e^{s\Delta}w\|_{L^2} + |e^{\gamma t} - e^{\gamma s}| \cdot \|e^{s\Delta}w\|_{L^2} \\ &\leq e^{\gamma t} \|(e^{(t-s)\Delta} - I)w\|_{L^2} + \gamma \|e^{\gamma \cdot}\|_{L^\infty(s,t)} |t-s| \cdot \|w\|_{L^2} \\ &\leq e^{\gamma T} (1 - e^{-\lambda_1(t-s)}) \|w\|_{L^2} + \gamma e^{\gamma T} |t-s| \cdot \|w\|_{L^2} \\ &\leq e^{\gamma T} \|w\|_{L^2} (\lambda_1 |t-s| + \gamma |t-s|) \leq C(T) |t-s|. \end{aligned} \quad \square$$

Lemma 4.22. *For $p \in [1, 2]$, there exists a $C > 0$ such that*

$$\mathbb{E}_n \left[\sup_{|t-s| \leq \delta} \|(\mathfrak{F}_B \eta)(t) - (\mathfrak{F}_B \eta)(s)\|_{L^2}^p \right] \leq C \delta^{\beta p} \|\xi\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0,T;L_{\mathcal{O}}^2)}^p.$$

Proof. Combining $H^\alpha \hookrightarrow L^2$, Lemma 4.19 and Lemma 4.20 we obtain

$$\|(\mathfrak{F}_B \eta)(t) - (\mathfrak{F}_B \eta)(s)\|_{L^2}^p \leq |t-s|^{\beta p} \|\mathfrak{F}_B\|_{C^\beta([0,T];H^\alpha)}^p \leq C_1 |t-s|^{\beta p} \sup_{t \in [0,T]} \|\xi(t)\|_{L^2}^p,$$

which leads to

$$\mathbb{E}_n \left[\sup_{|t-s| \leq \delta} \|(\mathfrak{F}_B \eta)(t) - (\mathfrak{F}_B \eta)(s)\|_{L^2}^p \right] \leq C_1 \delta^{\beta p} \mathbb{E}_n \left[\sup_{s \in [0,T]} \|\xi(s)\|_{L^2}^p \right] \leq C_2 \delta^{\beta p} \|\xi\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0,T;L_{\mathcal{O}}^2)}^p. \quad \square$$

Lemma 4.23. *For each $n \in \mathbb{N}$ and every $p \in [1, 2+\varepsilon_u]$, there exists $C = C(R, n, p, \varepsilon_u, T) > 0$*

such that

$$\sup_{\xi \in \mathcal{C}_R} \mathbb{E}_n \left[\sup_{t \in [0, T]} \|(\mathfrak{T}_n \xi)(t)\|_{L^2}^p \right] \leq C < \infty. \quad (4.15)$$

Proof. Let first $p = 2 + \varepsilon_u$, $\xi \in \mathcal{C}_R$ and $u_n = \mathfrak{T}_n \xi$. Since u_n can be taken apart into three addends we can treat them separately. Firstly we have

$$\mathbb{E}_n \left[\sup_{t \in [0, T]} \|e^{tB} u_0\|_{L^2_{\mathcal{O}}}^p \right] \leq e^{p\gamma T} \mathbb{E}_n \left[\|u_0\|_{L^2_{\mathcal{O}}}^p \right] \leq C_1.$$

As in the proof of Lemma 4.17 we can see that with Jensen

$$\begin{aligned} & \mathbb{E}_n \left[\sup_{t \in [0, T]} \left\| \int_0^s e^{(s-\sigma)B} \Psi_n^1(\sigma) \Psi_n^2(\sigma) \Psi_n^3(\sigma) \operatorname{div}(\xi(\sigma) \nabla v(\sigma)) \, d\sigma \right\|_{L^2(\mathcal{O})}^p \right] \\ & \leq \mathbb{E}_n \left[\sup_{t \in [0, T]} \|\xi\|_{L^2}^{2\frac{p}{4}} \right] \leq C_2 \|\xi\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2_{\mathcal{O}})}^{2\frac{p}{4}} \leq C_2 R^{\frac{p}{2}}. \end{aligned}$$

The third term will be handled via the Burkholder–Davis–Gundy inequality (Theorem 2.35)

$$\begin{aligned} \mathbb{E}_n \left[\sup_{t \in [0, T]} \left\| \int_0^t e^{(t-s)B} u_n(s) \, d\mathcal{W}(s) \right\|_{L^2}^p \right] & \leq C_3 \mathbb{E}_n \left[\left(\int_0^T \|e^{(T-s)B} u_n(s)\|_{L^2}^2 \, ds \right)^{\frac{p}{2}} \right] \\ & \leq C'_3 \int_0^T \mathbb{E}_n \left[\sup_{s \in [0, t]} \|u_n(s)\|_{L^2}^p \right] \, dt, \end{aligned}$$

and hence in total

$$\mathbb{E}_n \left[\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^p \right] \leq C_4 + \int_0^T \mathbb{E}_n \left[\sup_{s \in [0, t]} \|u_n(s)\|_{L^2}^p \right] \, dt,$$

which by Gronwall yields the statement. For $p < 2 + \varepsilon_u$ one simply uses Hölder on the probability space. $\square$

One might now expect us to prove that $\mathfrak{T}_n$ is compact, but we will not achieve that result. Instead we can show tightness of a family of laws which we will use later in our Schauder–Tychonoff fixed point theorem.

To prove this tightness we take the supremum $\sup_{|t-s| \leq \delta}$ over the three addends. For the stochastic integral, for which this is not straightforward, we make use of the Kolmogorov theorem:

Lemma 4.24. *For $p \in [1, 2 + \varepsilon_u]$, there exists a $C = C(T, p, \gamma, n, R, \varepsilon_u) > 0$ such that*

$$\mathbb{E}_n \left[\|\mathfrak{S}_B u_n(t) - \mathfrak{S}_B u_n(s)\|_{L^2}^p \right] \leq C |t - s|^{\frac{p}{2}}.$$

Proof. We first observe the identity for $s \leq t$

$$\mathfrak{S}_B u_n(t) - \mathfrak{S}_B u_n(s) = \int_s^t e^{(t-\sigma)B} u_n(\sigma) \, d\mathcal{W}(\sigma) + \left(e^{(t-s)B} - I \right) \mathfrak{S}_B u_n(s).$$

For the first term we see with Lemma 4.23

$$\begin{aligned}
\mathbb{E}_n \left[\sup_{t' \in [s, t]} \left\| \int_s^{t'} e^{(t'-\sigma)B} u_n(\sigma) d\mathcal{W}(\sigma) \right\|_{L^2}^p \right] &\leq C \mathbb{E}_n \left[\left(\int_s^t \overbrace{\|e^{(t-\sigma)B} u_n(\sigma)\|_{L^2}^2}^{\leq e^{2\gamma T} \|u_n(\sigma)\|_{L^2}^2} d\sigma \right)^{\frac{p}{2}} \right] \\
&\leq C_2 |t-s|^{\frac{p}{2}-1} \mathbb{E}_n \left[\int_s^t \|u_n(\sigma)\|_{L^2}^p d\sigma \right] \leq C_2 |t-s|^{\frac{p}{2}-1} \int_s^t \mathbb{E}_n \left[\sup_{\sigma' \in [0, \sigma]} \|u_n(\sigma')\|_{L^2}^p \right] d\sigma \\
&\leq C_3 |t-s|^{\frac{p}{2}}.
\end{aligned}$$

The second term can be handled by Minkowski, semigroup estimates, $1 - e^{-x} \leq x$, $e^x - 1 = e^x(1 - e^{-x})$

$$\begin{aligned}
\left\| (e^{(t-s)B} - I) \mathfrak{S}_B u_n(s) \right\|_{L^2} &\leq \left\| (e^{(t-s)\Delta} - I) \mathfrak{S}_B u_n(s) \right\|_{L^2} + \|e^{(t-s)\Delta} (e^{(t-s)\gamma} - 1) \mathfrak{S}_B u_n(s)\| \\
&\leq C_1 (1 - e^{-\lambda_1(t-s)}) \|\mathfrak{S}_B u_n(s)\|_{L^2} + (e^{(t-s)\gamma} - 1) \|\mathfrak{S}_B u_n(s)\|_{L^2} \leq C_2 |t-s| \|\mathfrak{S}_B u_n(s)\|_{L^2},
\end{aligned}$$

and with the Burkholder–Davis–Gundy inequality (Theorem 2.35) and Lemma 4.23

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|\mathfrak{S}_B u_n(s)\|_{L^2}^p \right] \leq C_3 \mathbb{E}_n \left[\left(\int_0^t \|u_n(s)\|_{L^2}^2 ds \right)^{\frac{p}{2}} \right] \leq C_3 T \mathbb{E}_n \left[\sup_{s \in [0, T]} \|u_n(s)\|_{L^2}^p \right] \leq C_4.$$

Hence in total we get

$$\mathbb{E}_n \left[\|\mathfrak{S}_B u_n(t) - \mathfrak{S}_B u_n(s)\|_{L^2}^p \right] \leq C_5 |t-s|^{\frac{p}{2}} + C_6 |t-s|^p \leq C_7 |t-s|^{\frac{p}{2}}. \quad \square$$

Corollary 4.25. *There exists a $C = C(T, \gamma, n, R, \varepsilon_u) > 0$ such that*

$$\mathbb{E}_n \left[\sup_{|t-s| \leq \delta} \|\mathfrak{S}_B u_n(t) - \mathfrak{S}_B u_n(s)\|_{L^2}^{2+\varepsilon_u} \right] \leq C \delta^{\frac{\varepsilon_u}{2}}.$$

Proof. Since $\varepsilon_u > 0$, the Kolmogorov Theorem (Theorem 2.34) applied to Lemma 4.24 (with $p = 2 + \varepsilon_u$) yields the claim. $\square$

Lemma 4.26 (Tightness of $\mathfrak{T}_n$). *The family of laws $\{\text{Law}(\mathfrak{T}_n \xi) \mid \xi \in \mathcal{C}_R\}$ is tight in $\mathcal{M}(C^0([0, T]; L^2(\mathcal{O})))$.*

Proof. Let $\mathcal{C}_R \subseteq \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ be defined as in (4.10). Let $\varepsilon > 0$ and choose

$$M_\varepsilon := 1 + \sqrt{\frac{2}{\varepsilon} \mathbb{E}_n [\|u_0\|_{H^\alpha}^2]} < \infty$$

(Even though u_0 is itself a random variable, M_ε is a well-defined deterministic constant, since the randomness is integrated out by $\mathbb{E}_n$; this is finite by Assumption 4.5(c).) and define the compact set

$$L_\varepsilon := \overline{\{u \in L^2(\mathcal{O}) : \|u\|_{H^\alpha} \leq M_\varepsilon\}}^{L^2(\mathcal{O})} \subset L^2(\mathcal{O}),$$

which is compact in $L^2(\mathcal{O})$ since $H^\alpha \xrightarrow{c} L^2(\mathcal{O})$ compactly for $\alpha > 0$ by the Rellich–Kondrachov theorem (see e.g. [AF03, Theorem 6.3]). Let $\omega(\delta, \varepsilon)$ be determined below. By

the Markov inequality (Lemma 2.9),

$$\begin{aligned}
\text{Law}(\mathfrak{T}_n \xi)(\{x \in C^0([0, T]; L^2) : x(0) \notin L_\varepsilon\}) &= \mathbb{P}_n(\mathfrak{T}_n \xi(0) \notin L_\varepsilon) \\
&\leq \mathbb{P}_n(\|\mathfrak{T}_n \xi(0)\|_{H^\alpha} > M_\varepsilon) \\
&\leq \frac{1}{M_\varepsilon^2} \mathbb{E}_n[\|\mathfrak{T}_n \xi(0)\|_{H^\alpha}^2] \\
&\leq \frac{1}{M_\varepsilon^2} \mathbb{E}_n[\|u_0\|_{H^\alpha}^2] < \frac{\varepsilon}{2}.
\end{aligned}$$

Recall that for $\eta = \Psi_n^1 \Psi_n^2 \Psi_n^3 \operatorname{div}(\xi \nabla v)$

$$\mathfrak{T}_n \xi(t) = e^{t(\Delta + \gamma I)} u_0 - \mathfrak{F}_B \eta(t) + \mathfrak{S}_B u_n(t) \quad \forall t \in [0, T].$$

Defining $\omega(\delta, \varepsilon)$ by

$$\omega(\delta, \varepsilon) := \frac{6C_1 \mathbb{E}_n[\|u_0\|_{L^2}]}{\varepsilon} \delta + \frac{6C_2 \|\xi\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}}{\varepsilon} \delta^\beta + \left(\frac{6C_3}{\varepsilon}\right)^{\frac{1}{2+\varepsilon_u}} \delta^{\frac{\varepsilon_u}{2(2+\varepsilon_u)}}$$

Jensen's inequality, Markov's inequality and Lemmas 4.21, 4.22 and Corollary 4.25 yield

$$\begin{aligned}
\text{Law}(\mathfrak{T}_n \xi)(\{x : \omega_x(\delta) > \omega(\delta, \varepsilon)\}) &= \mathbb{P}_n\left(\sup_{|t-s| \leq \delta} \|(\mathfrak{T}_n \xi)(t) - (\mathfrak{T}_n \xi)(s)\|_{L^2} > \omega(\delta, \varepsilon)\right) \\
&\leq \mathbb{P}_n\left(\sup_{|t-s| \leq \delta} \|e^{tB} u_0 - e^{sB} u_0\|_{L^2} > \frac{\omega(\delta, \varepsilon)}{3}\right) + \mathbb{P}_n\left(\sup_{|t-s| \leq \delta} \|\mathfrak{F}_B \eta(t) - \mathfrak{F}_B \eta(s)\|_{L^2} > \frac{\omega(\delta, \varepsilon)}{3}\right) \\
&\quad + \mathbb{P}_n\left(\sup_{|t-s| \leq \delta} \|\mathfrak{S}_B u_n(t) - \mathfrak{S}_B u_n(s)\|_{L^2} > \frac{\omega(\delta, \varepsilon)}{3}\right) \\
&\leq \frac{3}{\omega(\delta, \varepsilon)} \mathbb{E}_n\left[\sup_{|t-s| \leq \delta} \|e^{tB} u_0 - e^{sB} u_0\|_{L^2}\right] + \frac{3}{\omega(\delta, \varepsilon)} \mathbb{E}_n\left[\sup_{|t-s| \leq \delta} \|\mathfrak{F}_B \eta(t) - \mathfrak{F}_B \eta(s)\|_{L^2}\right] \\
&\quad + \left(\frac{3}{\omega(\delta, \varepsilon)}\right)^{2+\varepsilon_u} \mathbb{E}_n\left[\sup_{|t-s| \leq \delta} \|\mathfrak{S}_B u_n(t) - \mathfrak{S}_B u_n(s)\|_{L^2}^{2+\varepsilon_u}\right] \\
&\leq \frac{C_1}{\omega(\delta, \varepsilon)} \delta \mathbb{E}_n[\|u_0\|_{L^2}] + \frac{C_2}{\omega(\delta, \varepsilon)} \delta^\beta \|\xi\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} + \frac{C_3}{\omega(\delta, \varepsilon)^{2+\varepsilon_u}} \delta^{\frac{\varepsilon_u}{2}} < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}
\end{aligned}$$

for all $\delta > 0$, uniformly in $\xi \in \mathcal{C}_R$. □

Continuity

Next we will show the continuity of $\mathfrak{T}_n : \mathcal{C}_R \rightarrow \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$. Again fix $n \in \mathbb{N}$ and consider $\xi^1, \xi^2 \in \mathcal{C}_R$, i.e., $\xi^j \in \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ with $\|\xi^j\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2 \leq R$ for $j \in \{1, 2\}$. In order to prove the continuity of $\mathfrak{T}_n$, we will show that there exists a constant $C = C(n) > 0$ such that

$$\|\mathfrak{T}_n(\xi^1) - \mathfrak{T}_n(\xi^2)\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} \leq C \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{\frac{1}{4}}. \quad (4.16)$$

Definition 4.27. For $j \in \{1, 2\}$ let $(u_n^j, v^j) = (\mathfrak{T}_n(\xi^j), \mathfrak{R}(\xi^j))$ be the solution to the system (4.7) where ξ shall be replaced by ξ^j . We also define $\Psi_{n,j}^1$, $\Psi_{n,j}^2$ and $\Psi_{n,j}^3$ analogous to

Definition 4.8 by

$$\begin{aligned}\Psi_{n,j}^1(s) &= \psi_n(h^1(\xi^j, s)) = \psi_n \left(\sup_{\sigma \in [0,s]} \|\xi^j(\sigma)\|_{L^1(\mathcal{O})} \right) = \psi \left(\frac{1}{n} \|\xi^j\|_{L^\infty(0,s;L^1(\mathcal{O}))} \right), \\ \Psi_{n,j}^2(s) &= \psi_n(h^2(v^j, s)) = \psi_n \left(\sup_{\sigma \in [0,s]} \|\nabla v^j(\sigma)\|_{L^2(\mathcal{O})} \right) = \psi \left(\frac{1}{n} \|\nabla v^j\|_{L^\infty(0,s;L^2(\mathcal{O}))} \right), \\ \Psi_{n,j}^3(s) &= \psi_n(h^3(v^j, s)) = \psi_n \left(\int_0^s \|\nabla v^j(\sigma)\|_{H^1(\mathcal{O})}^2 d\sigma \right) = \psi \left(\frac{1}{n} \|\nabla v^j\|_{L^2(0,s;H^1(\mathcal{O}))}^2 \right).\end{aligned}$$

Remark 4.28. For these three functions, $a \geq b > 0$ and $\alpha \in (0, 1]$, $q \in (0, \frac{2}{\alpha})$ the following are true.

$$\begin{aligned}(i) \quad & \Psi_{n,j}^1(t)^a \sup_{s \in [0,t]} \|\xi^j(s)\|_{L^1}^b \leq (2n)^b, \quad (ii) \quad \Psi_{n,j}^1(t)^a \sup_{s \in [0,t]} \|\nabla v^j(s)\|_{L^2}^b \leq (2n)^b, \\ (iii) \quad & \int_0^t \Psi_{n,j}^3(s) \|\nabla v^j(s)\|_{H^1}^2 ds \leq 2n, \quad (iv) \quad \int_0^t \Psi_{n,j}^3(s)^q \|\nabla v^j(s)\|_{H^1}^{q\alpha} ds \leq \max(1, T)(2n)^{\frac{q\alpha}{2}}.\end{aligned}$$

Proof. Lemma 4.9. □

Lemma 4.29. *There exists a constant $C > 0$, independent of n , such that*

$$\begin{aligned}|\Psi_{n,1}^1(s) - \Psi_{n,2}^1(s)| &\leq \frac{1}{n} \|\psi'\|_{L^\infty} \|\xi^1 - \xi^2\|_{L^\infty(0,s;L^1(\mathcal{O}))}, \\ |\Psi_{n,1}^2(s) - \Psi_{n,2}^2(s)| &\leq \frac{1}{n} \|\psi'\|_{L^\infty} \|\nabla v^1 - \nabla v^2\|_{L^\infty(0,s;L^2(\mathcal{O}))}, \\ |\Psi_{n,1}^3(s) - \Psi_{n,2}^3(s)| &\leq \frac{C}{n} \|\psi'\|_{L^\infty} \|\nabla v^1 - \nabla v^2\|_{L^2(0,s;H^1(\mathcal{O}))}.\end{aligned}$$

Proof. Since ψ_n is continuous and differentiable on $\mathbb{R}$, the mean value theorem implies that for $\eta_1, \eta_2 \in \mathbb{R}$ with $\eta_1 < \eta_2$ there exists a $\zeta \in (\eta_1, \eta_2)$ such that

$$\frac{\psi_n(\eta_1) - \psi_n(\eta_2)}{\eta_1 - \eta_2} = \psi'_n(\zeta) = \frac{1}{n} \psi'(\zeta) \leq \frac{1}{n} \|\psi'\|_{L^\infty}.$$

With the fact that $\|\cdot\|_{L^\infty(0,s;L^1(\mathcal{O}))}$ is a norm, together with the Minkowski inequality, we get

$$\begin{aligned}|\psi_n(h^1(\xi^1, s)) - \psi_n(h^1(\xi^2, s))| &\leq \frac{1}{n} \|\psi'\|_{L^\infty} |h^1(\xi^1, s) - h^1(\xi^2, s)| \\ &= \frac{1}{n} \|\psi'\|_{L^\infty} \left| \|\xi^1\|_{L^\infty(0,s;L^1(\mathcal{O}))} - \|\xi^2\|_{L^\infty(0,s;L^1(\mathcal{O}))} \right| \\ &\leq \frac{1}{n} \|\psi'\|_{L^\infty} \|\xi^1 - \xi^2\|_{L^\infty(0,s;L^1(\mathcal{O}))}.\end{aligned}$$

The same calculation leads to the second inequality. For the third we can see that

$$\begin{aligned}|\psi_n(h^3(v^1, s)) - \psi_n(h^3(v^2, s))| &\leq \frac{1}{n} \|\psi'\|_{L^\infty} |h^3(v^1, s) - h^3(v^2, s)| \\ &= \frac{1}{n} \|\psi'\|_{L^\infty} \left| \|\nabla v^1\|_{L^2(0,s;H^1(\mathcal{O}))}^2 - \|\nabla v^2\|_{L^2(0,s;H^1(\mathcal{O}))}^2 \right| \\ &= \frac{1}{n} \|\psi'\|_{L^\infty} \left(\|\nabla v^1\|_{L_s^2 H^1} + \|\nabla v^2\|_{L_s^2 H^1} \right) \left| \|\nabla v^1\|_{L_s^2 H^1} - \|\nabla v^2\|_{L_s^2 H^1} \right| \\ &\leq \frac{1}{n} \|\psi'\|_{L^\infty} \|\nabla v^1 - \nabla v^2\|_{L_s^2 H^1}\end{aligned}$$

where $C = \|\nabla v^1\|_{L_s^2 H^1} + \|\nabla v^2\|_{L_s^2 H^1}$ is finite and independent of n because by the maximal regularity estimate of Lemma 2.30, $v^1, v^2 \in L^2(0, s; H^1(\mathcal{O}))$ with bounds only depending on ξ and not on n . $\square$

We now want to estimate the terms $\|\nabla v^1 - \nabla v^2\|_{L_s^\infty L_{\mathcal{O}}^2}$ and $\|\nabla v^1 - \nabla v^2\|_{L_s^2 H_{\mathcal{O}}^1}$. For that we first need to control $\|v^1 - v^2\|_{L^2}$ via Gronwall's inequality.

Lemma 4.30. *For $d \in \{1, 2, 3, 4\}$ there exists a constant $C = C(T, d, u_0) > 0$ such that for $w = v^1 - v^2$ we have $\mathbb{P}_n$ -a.s. for all $t \in [0, T]$*

$$\|w(t)\|_{L^2} \leq C \left(\int_0^t \left(\|\xi^1(s) - \xi^2(s)\|_{L^2} + \|\nabla w(s)\|_{L^2}^{\frac{d}{2}} \right) ds \right)^{\frac{4}{4+d}}.$$

Proof. Let $w = v^1 - v^2$. Since $v^j \in L_T^\infty H_{\mathcal{O}}^1 \cap L_T^2 H_{\mathcal{O}}^2$, we can find a $C_1 > 0$ such that $\|w(t)\|_{H^1} \leq \|v^1(t)\|_{H^1} + \|v^2(t)\|_{H^1} \leq C_1$ for a.e. $t \in [0, T]$. Using $\partial_t w = \Delta w - (\xi^1 v^1 - \xi^2 v^2)$, we can calculate with $\rho = \frac{1}{2} - \frac{d}{8}$ and $\|w\|_{L^2}^{4\rho} \leq C_1^{2\rho} \|w\|_{L^2}^{2\rho}$ that $\mathbb{P}_n$ -a.s.

$$\begin{aligned} \frac{1}{2} \partial_t \|w\|_{L^2}^2 &= \int_{\mathcal{O}} w \Delta w - \int_{\mathcal{O}} w (\xi^1 v^1 - \xi^2 v^2) = -\|\nabla w\|_{L^2}^2 - \int_{\mathcal{O}} w ((\xi^1 - \xi^2) v^1 + \xi^2 (v^1 - v^2)) \\ &\leq -\int_{\mathcal{O}} v^1 w (\xi^1 - \xi^2) - \int_{\mathcal{O}} \xi^2 |w|^2 \leq \|v^1\|_{L^4} \|w\|_{L^4} \|\xi^1 - \xi^2\|_{L^2} + \|\xi^2\|_{L^2} \|w\|_{L^4}^2 \\ &\leq C_2^2 \|\nabla v^1\|_{L^2}^{\frac{d}{4}} \|v^1\|_{L^2}^{1-\frac{d}{4}} \|\nabla w\|_{L^2}^{\frac{d}{4}} \|w\|_{L^2}^{1-\frac{d}{4}} \|\xi^1 - \xi^2\|_{L^2} + C_2^2 \|\xi^2\|_{L^2} \|\nabla w\|_{L^2}^{\frac{d}{2}} \|w\|_{L^2}^{2-\frac{d}{2}} \\ &\leq \frac{1}{2} C_3(T, d) \left(\|\xi^1 - \xi^2\|_{L^2} + \|\nabla w\|_{L^2}^{\frac{d}{2}} \right) \|w\|_{L^2}^{2\rho}, \end{aligned}$$

which then with $w(0) \equiv 0$ and the nonlinear Gronwall inequality (Lemma 2.18(ii), with $\beta_1 \equiv 0$, $y_0 = 0$ and $\rho < 1$) yields for $C_4(T, d) = C_3(T, d) \left(\frac{1}{2} + \frac{d}{8} \right)$

$$\|w(t)\|_{L^2}^2 \leq \left(C_4(T, d) \int_0^t \left(\|\xi^1(s) - \xi^2(s)\|_{L^2} + \|\nabla w(s)\|_{L^2}^{\frac{d}{2}} \right) ds \right)^{\frac{8}{4+d}} \quad \mathbb{P}_n\text{-a.s.}$$

Taking the square root shows the claim. ($\mathbb{P}_n$ -a.s. arises from estimating $\|\xi(s)\|_{L^2}$.) $\square$

Corollary 4.31. *For $d \in \{1, 2, 3, 4\}$ there exists a constant $C = C(T, d) > 0$ such that $\mathbb{P}_n$ -a.s. for all $t \in [0, T]$*

$$\|v^1(t) - v^2(t)\|_{L^2}^2 \leq C \int_0^t \left(\|\xi^1(s) - \xi^2(s)\|_{L^2}^{\frac{8}{4+d}} + \|\nabla v^1(s) - \nabla v^2(s)\|_{L^2}^{\frac{4d}{4+d}} \right) ds.$$

Proof. Because $\frac{8}{4+d} \geq 1$ for $d \in \{1, 2, 3, 4\}$, the function $x \mapsto |x|^{\frac{8}{4+d}}$ is convex. Hence applying two different Jensen inequalities (Corollary 2.4 and Lemma 2.1) to the result of Lemma 4.30 proves the claim. $\square$

Lemma 4.32. *For any measurable non-negative function y , we obtain the following identity*

$$\int_0^t (t-s)^{-\frac{1}{2}} \int_0^s y(\sigma) d\sigma ds = 2 \int_0^t (t-s)^{\frac{1}{2}} y(s) ds.$$

Proof. Fubini ($0 \leq \sigma \leq s \leq t$) shows that

$$\begin{aligned} \int_0^t (t-s)^{-\frac{1}{2}} \int_0^s y(\sigma) d\sigma ds &= \int_0^t \int_0^s (t-s)^{-\frac{1}{2}} y(\sigma) d\sigma ds = \int_0^t \int_\sigma^t (t-s)^{-\frac{1}{2}} y(\sigma) ds d\sigma \\ &= \int_0^t \int_0^{t-\sigma} s^{-\frac{1}{2}} ds y(\sigma) d\sigma = 2 \int_0^t (t-\sigma)^{\frac{1}{2}} y(\sigma) d\sigma. \quad \square \end{aligned}$$

Lemma 4.33. *For a non-negative deterministic function $f \in L^1(0, T)$ and a progressively measurable, real- (or Banach-space-)valued process X with $\int_0^T \mathbb{E}[|X(\sigma)|^2] d\sigma < \infty$ (so that Tonelli's theorem applies below) we get*

$$\mathbb{E} \left[\sup_{s \in [0, t]} \int_0^s f(s-\sigma) |X(\sigma)|^2 d\sigma \right] \leq \int_0^t f(t-s) \mathbb{E} \left[\sup_{\sigma \in [0, s]} |X(\sigma)|^2 \right] ds.$$

Proof. Straight forward computation shows

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} \int_0^s f(s-\sigma) |X(\sigma)|^2 d\sigma \right] &\leq \mathbb{E} \left[\sup_{s \in [0, t]} \int_0^s f(s-\sigma) \sup_{r \in [0, \sigma]} |X(r)|^2 d\sigma \right] \\ &= \mathbb{E} \left[\sup_{s \in [0, t]} \int_0^s f(\sigma) \sup_{r \in [0, s-\sigma]} |X(r)|^2 d\sigma \right] \leq \mathbb{E} \left[\int_0^t f(\sigma) \sup_{r \in [0, t-\sigma]} |X(r)|^2 d\sigma \right] \\ &= \mathbb{E} \left[\int_0^t f(t-\sigma) \sup_{r \in [0, \sigma]} |X(r)|^2 d\sigma \right] = \int_0^t f(t-\sigma) \mathbb{E} \left[\sup_{r \in [0, \sigma]} |X(r)|^2 \right] d\sigma. \quad \square \end{aligned}$$

Lemma 4.34. *For $d = 1$, there exists $C = C(T) > 0$ such that*

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|\nabla v^1(s) - \nabla v^2(s)\|_{L^2}^2 \right] \leq C \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}. \quad (4.17)$$

Proof. The identity from (4.9) applied to v^j gives us

$$v^j(t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \xi^j(s) v^j(s) ds.$$

Hence we can write the difference in the following way

$$v^1(t) - v^2(t) = - \int_0^t e^{(t-s)\Delta} \left((\xi^1(s) - \xi^2(s)) v^1(s) + \xi^2(s) (v^1(s) - v^2(s)) \right) ds$$

Using the semigroup inequality in Lemma 2.19(iii), $e^{-\lambda_1 t} \leq 1 \leq T^{\frac{1}{2}} (t-s)^{-\frac{1}{2}}$, the Minkowski and the Hölder inequality, we can calculate

$$\begin{aligned} \|\nabla v^1(t) - \nabla v^2(t)\|_{L^2} &\leq \int_0^t \left\| \nabla e^{(t-s)\Delta} \left((\xi^1(s) - \xi^2(s)) v^1(s) + \xi^2(s) (v^1(s) - v^2(s)) \right) \right\|_{L^2} ds \\ &\leq C \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \right) \left\| (\xi^1(s) - \xi^2(s)) v^1(s) + \xi^2(s) (v^1(s) - v^2(s)) \right\|_{L^2} ds \\ &\leq C(T) \int_0^t (t-s)^{-\frac{1}{2}} \left\| (\xi^1(s) - \xi^2(s)) v^1(s) + \xi^2(s) (v^1(s) - v^2(s)) \right\|_{L^2} ds \\ &\leq C(T) \int_0^t (t-s)^{-\frac{1}{2}} \left(\|\xi^1(s) - \xi^2(s)\|_{L^2} \|v^1(s)\|_{L^\infty} + \|\xi^2(s)\|_{L^2} \|v^1(s) - v^2(s)\|_{L^\infty} \right) ds \end{aligned}$$

In $d = 1$ we have the embedding $H^1 \hookrightarrow L^\infty$ and hence we can find a constant $C > 0$ such that we can bound $\|v^1(s)\|_{L^\infty}$ from above by C . $\|\xi^2(s)\|_{L^2}$ is bounded $\mathbb{P}_n$ -a.s. for all $s \in [0, T]$ as well. Using

$$\int_0^t (t-s)^{-\frac{1}{2}} ds = \int_0^t s^{-\frac{1}{2}} ds = 2\sqrt{t} \leq 2\sqrt{T},$$

the Hölder inequality yields

$$\begin{aligned} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|\xi^1(s) - \xi^2(s)\|_{L^2} ds \right)^2 &= \left(\int_0^t (t-s)^{-\frac{1}{4}} (t-s)^{-\frac{1}{4}} \|\xi^1(s) - \xi^2(s)\|_{L^2} ds \right)^2 \\ &\leq \int_0^t (t-s)^{-\frac{1}{2}} ds \cdot \int_0^t (t-s)^{-\frac{1}{2}} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 ds \\ &\leq 2\sqrt{T} \sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 \int_0^t (t-s)^{-\frac{1}{2}} ds \\ &\leq 4T \sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2. \end{aligned}$$

For the second term we obtain again with $H^1 \hookrightarrow L^\infty$ (for $d = 1$), Corollary 4.31 and Lemma 4.32 that for $w = v^1 - v^2$

$$\begin{aligned} \left(\int_0^t (t-s)^{-\frac{1}{2}} \|w(s)\|_{L^\infty} ds \right)^2 &\leq 2\sqrt{T} \int_0^t (t-s)^{-\frac{1}{2}} \|w(s)\|_{H^1}^2 ds \\ &\leq 2\sqrt{T} \int_0^t (t-s)^{-\frac{1}{2}} \|w(s)\|_{L^2}^2 ds + 2\sqrt{T} \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla w(s)\|_{L^2}^2 ds \\ &\leq 2\sqrt{T} \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla w(s)\|_{L^2}^2 ds + C_1(T) \int_0^t (t-s)^{-\frac{1}{2}} \int_0^s \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2}^{\frac{8}{4+d}} d\sigma ds \\ &\quad + C_1(T) \int_0^t (t-s)^{-\frac{1}{2}} \int_0^s \|\nabla w(\sigma)\|_{L^2}^{\frac{4d}{4+d}} d\sigma ds \\ &\leq C_2(T) \int_0^t \left[(t-s)^{-\frac{1}{2}} \|\nabla w(s)\|_{L^2}^2 + (t-s)^{\frac{1}{2}} \|\nabla w(s)\|_{L^2}^{\frac{4d}{4+d}} \right] ds \\ &\quad + \int_0^t (t-s)^{\frac{1}{2}} \|\xi^1(s) - \xi^2(s)\|_{L^2}^{\frac{8}{4+d}} ds. \end{aligned}$$

Where the last addend can be bounded from above via the Hölder inequality (Theorem 2.6) and the fact that $\|\xi^1(s) - \xi^2(s)\|_{L^2}$ is bounded $\mathbb{P}_n$ -a.s. for every $s \in [0, T]$

$$\begin{aligned} \left(\int_0^t (t-s)^{\frac{1}{2}} \|\xi^1(s) - \xi^2(s)\|_{L^2}^{\frac{8}{4+d}} ds \right)^2 &\leq \int_0^t (t-s) ds \cdot \int_0^t \|\xi^1(s) - \xi^2(s)\|_{L^2}^{\frac{16}{4+d}} ds \\ &\leq T \cdot C^{\frac{16}{4+d}-2} \int_0^t \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 ds \leq C_2 \sup_{s \in [0, T]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2. \end{aligned}$$

This all together leads to

$$\begin{aligned}
& \mathbb{E}_n \left[\sup_{s \in [0, t]} \|\nabla w(s)\|_{L^2}^2 \right] \\
& \leq C_3 \mathbb{E}_n \left[\sup_{s \in [0, t]} \left(\left(\int_0^s (s - \sigma)^{-\frac{1}{2}} \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2} d\sigma \right)^2 + \left(\int_0^s (s - \sigma)^{-\frac{1}{2}} \|w(\sigma)\|_{L^\infty} d\sigma \right)^2 \right) \right] \\
& \leq C_4 \mathbb{E}_n \left[\sup_{s \in [0, T]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 \right] + C_5 \mathbb{E}_n \left[\sup_{s \in [0, T]} \|\xi^1(s) - \xi^2(s)\|_{L^2} \right] \\
& \quad + C_6 \mathbb{E}_n \left[\sup_{s \in [0, t]} \int_0^s \left((s - \sigma)^{-\frac{1}{2}} \|\nabla w(\sigma)\|_{L^2}^2 + (s - \sigma)^{\frac{1}{2}} \|\nabla w(\sigma)\|_{L^2}^{\frac{2d}{4+d}} \right) d\sigma \right] \\
& \leq C_4 \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2 + C_5 \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} \\
& \quad + C_6 \int_0^t (t - s)^{-\frac{1}{2}} \mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \|\nabla w(\sigma)\|_{L^2}^2 \right] ds + \int_0^t (t - s)^{\frac{1}{2}} \left(\mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \|\nabla w(\sigma)\|_{L^2}^2 \right] \right)^{\frac{2d}{4+d}} ds,
\end{aligned}$$

where all constants depend on T . Since $\frac{2d}{4+d} < 1$ for $d \leq 3$, Gronwall's inequality (Lemma 2.18) shows that

$$\begin{aligned}
\mathbb{E}_n \left[\sup_{s \in [0, t]} \|\nabla w(s)\|_{L^2}^2 \right] & \leq C_7 \left(\|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2 + \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} \right) \\
& \quad \times e^{\int_0^t (t-s)^{-\frac{1}{2}} ds} \left(1 - \frac{2d}{4+d} \right) \int_0^t (t-s)^{\frac{1}{2}} e^{-(1-\frac{2d}{4+d}) \int_0^s (s-\sigma)^{-\frac{1}{2}} d\sigma} ds \\
& \leq C_8(T, d) \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}. \quad \square
\end{aligned}$$

Lemma/Corollary 4.35. *For $d = 1$, there exists a constant $C = C(T) > 0$ such that*

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|v^1(s) - v^2(s)\|_{L^2}^2 \right] \leq C \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}.$$

Proof. Using the semigroup contraction (Lemma 2.19(i)) we obtain for $w = v^1 - v^2$ that

$$\begin{aligned}
\|w(s)\|_{L^2}^2 & = \left\| \int_0^s e^{(s-\sigma)\Delta} (\xi^1(\sigma)v^1(\sigma) - \xi^2(\sigma)v^2(\sigma)) d\sigma \right\|_{L^2}^2 \\
& \leq \int_0^s \left\| e^{(s-\sigma)\Delta} (\xi^1(\sigma)v^1(\sigma) - \xi^2(\sigma)v^2(\sigma)) \right\|_{L^2}^2 d\sigma \\
& \leq \int_0^s \left\| (\xi^1(\sigma) - \xi^2(\sigma))v^1(\sigma) + \xi^2(\sigma)(v^1(\sigma) - v^2(\sigma)) \right\|_{L^2}^2 d\sigma \\
& \leq 2 \int_0^s \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2}^2 \|v^1(\sigma)\|_{L^\infty}^2 + \|\xi^2(\sigma)\|_{L^2}^2 \|v^1(\sigma) - v^2(\sigma)\|_{L^\infty}^2 d\sigma \\
& \leq C_1(T) \sup_{\sigma \in [0, s]} \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2}^2 + C_2 \int_0^s \|w(\sigma)\|_{H^1}^2 d\sigma \\
& \leq C_1(T) \sup_{\sigma \in [0, s]} \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2}^2 + C_3 \sup_{\sigma \in [0, s]} \|\nabla w(\sigma)\|_{L^2}^2 + C_2 \int_0^s \|w(\sigma)\|_{L^2}^2 d\sigma.
\end{aligned}$$

With Lemma 4.34 and Lemma 4.33 we can see that

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|w(s)\|_{L^2}^2 \right] \leq C_4(T) \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} + C_5 \int_0^t \mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \|w(\sigma)\|_{L^2}^2 \right] ds,$$

which then with the Gronwall inequality (Lemma 2.18) leads directly to the claim. $\square$

Lemma 4.36. *For $d = 1$, there exists $C = C(T) > 0$ such that*

$$\mathbb{E}_n \left[\int_0^t \|\nabla v^1(s) - \nabla v^2(s)\|_{H^1}^2 ds \right] \leq C \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}.$$

Proof. Because of the maximal regularity (Lemma 2.30) we can estimate

$$\begin{aligned} \int_0^t \|\nabla w(s)\|_{H^1}^2 ds &\leq \|w\|_{L_t^2 H_{\mathcal{O}}^2}^2 \leq C_1 \|\xi^1 v^1 - \xi^2 v^2\|_{L_t^2 L_{\mathcal{O}}^2}^2 \\ &\leq C_2 \int_0^t \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 ds + C_3 \int_0^t \|v^1(s) - v^2(s)\|_{L^\infty}^2 ds \\ &\leq C_4(T) \sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 + C_3 \int_0^t \|w(s)\|_{L^2}^2 ds + C_3 \int_0^t \|\nabla w(s)\|_{L^2}^2 ds \\ &\leq C_5(T) \sup_{s \in [0, t]} \left(\|\xi^1(s) - \xi^2(s)\|_{L^2}^2 + \|w(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2 \right). \end{aligned}$$

Applying Lemma 4.34 and Lemma 4.35 we obtain the claim. $\square$

Corollary 4.37. *For $d = 1$ and every $q \geq 1$, there exists a constant $C = C(T, n, q, \mathcal{O})$, such that for all $t \in [0, T]$*

$$\sum_{i=1}^3 \mathbb{E}_n \left[|\Psi_{n,1}^i(t) - \Psi_{n,2}^i(t)|^q \right] \leq C \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{\frac{1}{2}}.$$

Proof. Obviously $|\Psi_{n,1}^i(t) - \Psi_{n,2}^i(t)| \in [0, 1]$ and hence

$$\mathbb{E}_n \left[|\Psi_{n,1}^i(t) - \Psi_{n,2}^i(t)|^q \right]^2 \leq \mathbb{E}_n \left[|\Psi_{n,1}^i(t) - \Psi_{n,2}^i(t)|^{2q} \right] \leq \mathbb{E}_n \left[|\Psi_{n,1}^i(t) - \Psi_{n,2}^i(t)|^2 \right].$$

This can then be further estimated in the following way

$$\begin{aligned} \mathbb{E}_n \left[|\Psi_{n,1}^1(t) - \Psi_{n,2}^1(t)|^2 \right] &\leq C_1 \mathbb{E}_n \left[\|\xi^1 - \xi^2\|_{L_T^\infty L_{\mathcal{O}}^1}^2 \right] \leq C_1 \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2, \\ \mathbb{E}_n \left[|\Psi_{n,1}^2(t) - \Psi_{n,2}^2(t)|^2 \right] &\leq C_2 \mathbb{E}_n \left[\|\nabla(v^1 - v^2)\|_{L_T^\infty L_{\mathcal{O}}^2}^2 \right] \leq C_2' \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2, \\ \mathbb{E}_n \left[|\Psi_{n,1}^3(t) - \Psi_{n,2}^3(t)|^2 \right] &\leq C_3 \mathbb{E}_n \left[\|\nabla(v^1 - v^2)\|_{L_T^2 H_{\mathcal{O}}^1}^2 \right] \leq C_3' \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2, \end{aligned}$$

where we used Lemmas 4.29, 4.34 and 4.36. $\square$

We now want to investigate the difference $\phi(t) := \mathfrak{T}_n \xi^1(t) - \mathfrak{T}_n \xi^2(t) = u_n^1(t) - u_n^2(t)$ for $\xi^1, \xi^2 \in \mathcal{C}_R$. Using (4.8) we can observe that

$$\phi(t) = I_1(t) + I_2(t) + I_3(t), \tag{4.18}$$

for

$$\begin{aligned} I_1(t) &:= - \int_0^t \Psi_{n,1}^2(s) \Psi_{n,1}^3(s) e^{(t-s)B} \operatorname{div} \left[(\xi^1 \Psi_{n,1}^1 - \xi^2 \Psi_{n,2}^1) \nabla v^1 \right] (s) \, ds, \\ I_2(t) &:= - \int_0^t \Psi_{n,2}^1(s) e^{(t-s)B} \operatorname{div} \left[\xi^2 (\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2) \right] (s) \, ds, \\ I_3(t) &:= \int_0^t e^{(t-s)B} \phi(s) \, d\mathcal{W}(s). \end{aligned}$$

For the term I_1 we will make use of certain stopping times.

Lemma 4.38. *For I_1 there exists a constant $C = C(T, n, \rho, q, \varepsilon, \delta) > 0$ such that*

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_1(s)\|_{L^2}^2 \right] \leq C \left(\|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} + \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{2(1-\frac{\delta}{\varepsilon})} \right)$$

for all $t \in [0, T]$ for some $\delta \in (0, \frac{1}{2})$ and $\varepsilon \in (\frac{1}{2}, \frac{3}{4})$. (No further relation between δ and ε is needed: since $\delta < \frac{1}{2} < \varepsilon$ automatically, the hypothesis $\delta < \varepsilon$ of Lemma 2.15, used below, is already satisfied.)

Proof. Using Minkowski's inequality for integrals, the semigroup divergence smoothing, L^2 product bound (Lemma 2.16) with $\delta \in (0, \frac{1}{2})$ and $\rho = \delta + \frac{1}{2} \in (\frac{1}{2}, 1)$, we obtain

$$\begin{aligned} \|I_1(s)\|_{L^2} &\leq \int_0^s \Psi_{n,1}^2(\sigma) \Psi_{n,1}^3(\sigma) \left\| e^{(s-\sigma)B} \operatorname{div} \left[(\xi^1 \Psi_{n,1}^1 - \xi^2 \Psi_{n,2}^1) \nabla v^1 \right] (\sigma) \right\|_{L^2} \, d\sigma \\ &\leq \int_0^s \Psi_{n,1}^2(\sigma) \Psi_{n,1}^3(\sigma) e^{(s-\sigma)\gamma} r(s-\sigma) \left\| (\xi^1 \Psi_{n,1}^1 - \xi^2 \Psi_{n,2}^1) \nabla v^1 (\sigma) \right\|_{L^2} \, d\sigma \\ &\leq e^{\gamma s} \sup_{\sigma \in [0, s]} \|(\xi^1 \Psi_{n,1}^1 - \xi^2 \Psi_{n,2}^1)(\sigma)\|_{H^{-\delta}} \int_0^s r(s-\sigma) \Psi_{n,1}^2(\sigma) \Psi_{n,1}^3(\sigma) \|\nabla v^1(\sigma)\|_{H^\rho} \, d\sigma \\ &\leq e^{\gamma s} \sup_{\sigma \in [0, s]} \|(\xi^1 \Psi_{n,1}^1 - \xi^2 \Psi_{n,2}^1)(\sigma)\|_{H^{-\delta}} \left(\int_0^s r(s-\sigma)^{q'} \, d\sigma \right)^{\frac{1}{q'}} \\ &\quad \times \left(\int_0^s \left(\Psi_{n,1}^2(\sigma) \Psi_{n,1}^3(\sigma) \|\nabla v^1(\sigma)\|_{H^\rho} \right)^q \, d\sigma \right)^{\frac{1}{q}} \end{aligned}$$

for some $q \in (2, \frac{2}{\rho})$ (hence $q' < 2$ and the integral over r is finite). The last integral can be bounded via Lemma 2.14 and the cutoffs in the following way

$$\begin{aligned} &\int_0^s \left(\Psi_{n,1}^2(\sigma) \Psi_{n,1}^3(\sigma) \|\nabla v^1(\sigma)\|_{H^\rho} \right)^q \, d\sigma \\ &\leq \sup_{\sigma \in [0, s]} \Psi_{n,1}^2(\sigma)^q \|\nabla v^1(\sigma)\|_{L^2}^{q(1-\rho)} \int_0^s \Psi_{n,1}^3(\sigma)^q \|\nabla v^1(\sigma)\|_{H^1}^{q\rho} \, d\sigma \\ &\leq (2n)^{q(1-\rho)} \cdot T(2n)^{\frac{q\rho}{2}}. \end{aligned}$$

Regarding the term with the supremum we introduce the following stopping times.

For $0 \leq s \leq \tau_1 := \min \left(\inf \{ \sigma > 0 : \|\xi^1(\sigma)\|_{L^1} \geq n \}, \inf \{ \sigma > 0 : \|\xi^2(\sigma)\|_{L^1} \geq n \}, T \right)$, we have $\|\xi^j(\sigma)\|_{L^1} \leq n$ for all $\sigma \in [0, s]$ and hence $\Psi_{n,j}^1(s) = 1$ for $j \in \{1, 2\}$. It Lemma 2.15 and

Minkowski show that

$$\|\xi^1(\sigma) - \xi^2(\sigma)\|_{H^{-\delta}} \leq \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^1}^{\frac{\delta}{\varepsilon}} \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \leq (2n)^{\frac{\delta}{\varepsilon}} \|\xi^1(\sigma) - \xi^2(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}}.$$

For $\tau_2 := \max\left((\inf\{\sigma > 0: \|\xi^1(\sigma)\|_{L^1} \geq 2n\} \wedge T), (\inf\{\sigma > 0: \|\xi^2(\sigma)\|_{L^1} \geq 2n\} \wedge T)\right)$ and $\tau_1 < s \leq \tau_2$, we have WLOG $\|\xi^1(\sigma)\|_{L^1} \leq 2n$ for all $\sigma \in [0, s]$ and hence

$$\begin{aligned} \|(\xi^1 \Psi_{n,1}^1 - \xi^2 \Psi_{n,2}^1)(\sigma)\|_{H^{-\delta}} &\leq \|\xi^1(\Psi_{n,1}^1 - \Psi_{n,2}^1)(\sigma)\|_{H^{-\delta}} + \|\Psi_{n,2}^1(\xi^1 - \xi^2)(\sigma)\|_{H^{-\delta}} \\ &\leq \|\xi^1(\sigma)\|_{L^1}^{\frac{\delta}{\varepsilon}} \|\xi^1(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} |\Psi_{n,1}^1 - \Psi_{n,2}^1(\sigma)| + \|\Psi_{n,2}^1(\xi^1 - \xi^2)(\sigma)\|_{L^1}^{\frac{\delta}{\varepsilon}} \|\xi^1 - \xi^2(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \\ &\leq (2n)^{\frac{\delta}{\varepsilon}} \|\xi^1(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} |\Psi_{n,1}^1 - \Psi_{n,2}^1(\sigma)| + \left(\|\xi^1(\sigma)\|_{L^1} + \|\Psi_{n,2}^1 \xi^2(\sigma)\|_{L^1}\right)^{\frac{\delta}{\varepsilon}} \|\xi^1 - \xi^2(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \\ &\leq (2n)^{\frac{\delta}{\varepsilon}} \|\xi^1(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \frac{C_1}{n} \sup_{r \in [0, \sigma]} \|\xi^1(r) - \xi^2(r)\|_{L^2}^{\frac{1}{2}} + (2n + 2n)^{\frac{\delta}{\varepsilon}} \|\xi^1 - \xi^2(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \end{aligned}$$

For $s > \tau_2$ we have $\Psi_{n,j}^1(s) = 0$ for $j \in \{1, 2\}$. Finally we get

$$\begin{aligned} \mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_1(s)\|_{L^2}^2 \right] &\leq C_2 \mathbb{E}_n \left[\sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 \right] + C_3 \mathbb{E}_n \left[\sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^{2(1-\frac{\delta}{\varepsilon})} \right] \\ &\leq C_2 \left(\mathbb{E}_n \left[\sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 \right] \right)^{\frac{1}{2}} + C_3 \left(\mathbb{E}_n \left[\sup_{s \in [0, t]} \|\xi^1(s) - \xi^2(s)\|_{L^2}^2 \right] \right)^{1-\frac{\delta}{\varepsilon}} \\ &\leq C_4 \left(\|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} + \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{2(1-\frac{\delta}{\varepsilon})} \right). \end{aligned}$$

where C_2, C_3, C_4 depend on $T, n, \rho, q, \varepsilon, \delta$. □

Lemma 4.39. *For I_2 there exists a constant $C = C(T, n, \rho) > 0$ such that*

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_2(s)\|_{L^2}^2 \right] \leq C \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{\frac{1}{2}},$$

for all $t \in [0, T]$.

Proof. Using

$$\begin{aligned} \|I_2(s)\|_{L^2} &\leq \int_0^s \Psi_{n,2}^1(\sigma) \left\| e^{(s-\sigma)B} \operatorname{div} \left[\xi^2(\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2) \right](\sigma) \right\|_{L^2} d\sigma \\ &\leq e^{\gamma s} \int_0^s r(s-\sigma) \Psi_{n,2}^1(\sigma) \left\| \left[\xi^2(\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2) \right](\sigma) \right\|_{L^2} d\sigma \\ &\leq e^{\gamma s} \left(\int_0^s r(s-\sigma)^{q'} d\sigma \right)^{\frac{1}{q'}} \left(\int_0^s \Psi_{n,2}^1(\sigma)^q \left\| \left[\xi^2(\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2) \right](\sigma) \right\|_{L^2}^q d\sigma \right)^{\frac{1}{q}}, \end{aligned}$$

with $q = \frac{2}{\rho}$ with $\rho = \delta + \frac{1}{2} \in (\frac{1}{2}, 1)$ for some $\delta \in (0, \frac{1}{2})$. Hence $q' < 2$, which again implies that the integral over r is finite and for some $\varepsilon \in (\frac{1}{2}, 1)$ we can use the product estimate of

Lemma 2.16 to obtain

$$\begin{aligned} & \left(\int_0^s \Psi_{n,2}^1(\sigma)^q \left\| \left[\xi^2(\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2) \right](\sigma) \right\|_{L^2}^q d\sigma \right)^{\frac{1}{q}} \\ & \leq \left(\int_0^s \Psi_{n,2}^1(\sigma)^{\frac{2}{\rho}} \left(\|\xi^2(\sigma)\|_{L^1}^{\frac{\delta}{\varepsilon}} \|\xi^2(\sigma)\|_{L^2}^{1-\frac{\delta}{\varepsilon}} \right)^{\frac{2}{\rho}} \left\| \left[\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2 \right](\sigma) \right\|_{H^\rho}^{\frac{2}{\rho}} d\sigma \right)^{\frac{\rho}{2}}. \end{aligned}$$

Since $\frac{\delta}{\varepsilon} < 1$, $\Psi_{n,2}^1 \|\xi^2\|_{L^1}^{\frac{\delta}{\varepsilon}}$ can be bounded by $2n$ and $\|\xi^2\|_{L^2}$ is $\mathbb{P}_n$ -a.s. bounded by R . Using

$$a_1 b_1 c_1 - a_2 b_2 c_2 = b_1 c_1 (a_1 - a_2) + a_2 c_1 (b_1 - b_2) + a_2 b_2 (c_1 - c_2)$$

we can see that by using $|\Psi_{n,j}^k| \leq 1$ (for notation simplicity we omit (σ))

$$\begin{aligned} & \int_0^s \left\| \Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2 \right\|_{H^\rho}^{\frac{2}{\rho}} \\ & \leq C_1 \int_0^s \left(\|\nabla(v^1 - v^2)\|_{H^\rho}^{\frac{2}{\rho}} + (\Psi_{n,1}^3 \|\nabla v^1\|_{H^\rho} |\Psi_{n,1}^2 - \Psi_{n,2}^2|)^{\frac{2}{\rho}} + (\Psi_{n,2}^2 \|\nabla v^1\|_{H^\rho} |\Psi_{n,1}^3 - \Psi_{n,2}^3|)^{\frac{2}{\rho}} \right) d\sigma \end{aligned} \quad (4.19)$$

The first addend in (4.19) can be estimated using $\|\cdot\|_{H^\rho} \leq C_2 \|\cdot\|_{L^2}^{1-\rho} \|\cdot\|_{H^1}^\rho \leq C_2 \|\cdot\|_{H^1}$, Minkowski, Bounds for $v^1, v^2 \in L_T^\infty H_\mathcal{O}^1 \cap L_T^2 H_\mathcal{O}^2$ (Lemma 4.11) in the following way (the exponent 2ρ will arise later in this proof)

$$\begin{aligned} \left(\int_0^s \|\nabla(v^1 - v^2)(\sigma)\|_{H^\rho}^{\frac{2}{\rho}} d\sigma \right)^{2\rho} & \leq \left(C_2 \int_0^s \|\nabla(v^1 - v^2)(\sigma)\|_{L^2}^{\frac{2}{\rho}-2} \|\nabla(v^1 - v^2)(\sigma)\|_{H^1}^2 d\sigma \right)^{2\rho} \\ & \leq \left(C_2 \sup_{\sigma \in [0,s]} \|\nabla(v^1 - v^2)(\sigma)\|_{L^2}^{\frac{2}{\rho}-2} \int_0^s \|\nabla(v^1 - v^2)(\sigma)\|_{H^1}^2 d\sigma \right)^{2\rho} \\ & \leq C_2^{2\rho} C_3^{4(1-\rho)} C_4^{1-2\rho} \int_0^s \|\nabla(v^1 - v^2)(\sigma)\|_{H^1}^2 d\sigma. \end{aligned}$$

Regarding the second addend in (4.19) we obtain

$$\begin{aligned} & \int_0^s \left((\Psi_{n,1}^3 \|\nabla v^1\|_{H^\rho} |\Psi_{n,1}^2 - \Psi_{n,2}^2|)(\sigma) \right)^{\frac{2}{\rho}} d\sigma \\ & \leq C_3 \sup_{\sigma \in [0,s]} |\Psi_{n,1}^2(\sigma) - \Psi_{n,2}^2(\sigma)|^{\frac{2}{\rho}} \int_0^s \Psi_{n,1}^3(\sigma)^{\frac{2}{\rho}} \|\nabla v^1(\sigma)\|_{H^1}^2 d\sigma \\ & \leq C_3 T^{\frac{2}{\rho}} (2n)^{\frac{1}{\rho}} \sup_{\sigma \in [0,s]} |\Psi_{n,1}^2(\sigma) - \Psi_{n,2}^2(\sigma)|^{\frac{2}{\rho}}. \end{aligned}$$

For the third addend in (4.19) with C_4 from Lemma 4.11 we obtain

$$\int_0^s \left((\Psi_{n,2}^2 \|\nabla v^1\|_{H^\rho} |\Psi_{n,1}^3 - \Psi_{n,2}^3|)(\sigma) \right)^{\frac{2}{\rho}} d\sigma \leq C_3 C_4^{\frac{2}{\rho}} T^{\frac{2}{\rho}} \sup_{\sigma \in [0,s]} |\Psi_{n,1}^3(\sigma) - \Psi_{n,2}^3(\sigma)|^{\frac{2}{\rho}}.$$

And hence in total using Jensen

$$\begin{aligned}
\left(\mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_2(s)\|_{L^2}^2 \right] \right)^2 &\leq C_5 \mathbb{E}_n \left[\sup_{s \in [0, t]} \left(\int_0^s \left\| \left[\Psi_{n,1}^2 \Psi_{n,1}^3 \nabla v^1 - \Psi_{n,2}^2 \Psi_{n,2}^3 \nabla v^2 \right] (\sigma) \right\|_{H^\rho}^{\frac{2}{\rho}} d\sigma \right)^{2\rho} \right] \\
&\leq C_6 \mathbb{E}_n \left[\sup_{s \in [0, t]} \left(\int_0^s \|\nabla(v^1 - v^2)(\sigma)\|_{H^1}^2 d\sigma + |\Psi_{n,1}^2(s) - \Psi_{n,2}^2(s)|^4 + |\Psi_{n,1}^3(s) - \Psi_{n,2}^3(s)|^4 \right) \right] \\
&\leq C_7 \cdot 3 \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)},
\end{aligned}$$

where we used the trivial embedding $H^\rho(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ (immediate for $\rho \geq 0$, since $H^\rho(\mathcal{O}) \subseteq H^0(\mathcal{O}) = L^2(\mathcal{O})$) together with the bounds established above. $\square$

Lemma 4.40. *For I_3 there exists a constant $C = C(T) > 0$ such that*

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_3(s)\|_{L_{\mathcal{O}}^2}^2 \right] \leq C \int_0^t \mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \|\phi(\sigma)\|_{L_{\mathcal{O}}^2}^2 \right] ds,$$

for all $t \in [0, T]$.

Proof. Using

$$\|e^{(s-\sigma)B} \phi(\sigma)\|_{L_{\mathcal{O}}^2} \leq e^{\gamma s} \|e^{(s-\sigma)\Delta} \phi(\sigma)\|_{L_{\mathcal{O}}^2} \leq e^{\gamma s} \|\phi(\sigma)\|_{L_{\mathcal{O}}^2},$$

we can see that with the Burkholder–Davis–Gundy inequality (Theorem 2.35)

$$\begin{aligned}
\mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_3(s)\|_{L_{\mathcal{O}}^2}^2 \right] &= \mathbb{E}_n \left[\left\| \int_0^s e^{(s-\sigma)B} \phi(\sigma) dW(\sigma) \right\|_{L_{\mathcal{O}}^2}^2 \right] \leq C_1 \mathbb{E}_n \left[\int_0^t \|e^{(t-s)B} \phi(s)\|_{L_{\mathcal{O}}^2}^2 ds \right] \\
&\leq C_2 \mathbb{E}_n \left[\int_0^t \|\phi(s)\|_{L_{\mathcal{O}}^2}^2 ds \right] \leq C_2 \int_0^t \mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \|\phi(\sigma)\|_{L_{\mathcal{O}}^2}^2 \right] ds. \quad \square
\end{aligned}$$

Lemma 4.41 (Continuity of $\mathfrak{T}_n$). *Let (u_0, v_0) fulfill Assumption 4.5. Then for any $R > 0$, the mapping $\mathfrak{T}_n: \mathcal{C}_R \rightarrow \mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O}))$ with $\mathcal{C}_R$ from (4.10) is Hölder continuous with exponent $\frac{1}{4}$. (R enters, throughout the proof below, only implicitly: the restriction $\xi^1, \xi^2 \in \mathcal{C}_R$ bounds $\|\xi^j\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)} \leq \sqrt{R}$, which is used in the constants of Lemmas 4.38 and 4.39 feeding into I_1, I_2 below.*

Proof. Since $\phi = I_1 + I_2 + I_3$ we can see that

$$\begin{aligned}
\mathbb{E}_n \left[\sup_{s \in [0, t]} \|\phi(s)\|_{L_{\mathcal{O}}^2}^2 \right] &\leq 3 \sum_{i=1}^3 \mathbb{E}_n \left[\sup_{s \in [0, t]} \|I_i(s)\|_{L_{\mathcal{O}}^2}^2 \right] \\
&\leq C_1 \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{\frac{1}{2}} + C_2 \int_0^t \mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \|\phi(\sigma)\|_{L_{\mathcal{O}}^2}^2 \right] ds,
\end{aligned}$$

and hence with Gronwall ($\phi(0) = u_0 - u_0 = 0$)

$$\|\mathfrak{T}_n \xi^1 - \mathfrak{T}_n \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^2 = \mathbb{E}_n \left[\sup_{t \in [0, T]} \|\mathfrak{T}_n \xi^1(t) - \mathfrak{T}_n \xi^2(t)\|_{L_{\mathcal{O}}^2}^2 \right] \leq C_3 \|\xi^1 - \xi^2\|_{\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L_{\mathcal{O}}^2)}^{\frac{1}{2}}. \quad \square$$

Existence of a fixed point

Since we do not know if $\mathfrak{T}_n$ is compact on some space and hence can not use Schauder's fixed point theorem, we will have to create our own Schauder–Tychonoff type fixed point theorem.

Lemma 4.42 (Step (a): Tightness of the Picard iteration). *For each $n \in \mathbb{N}$, define the recurrence sequence $\{(u_n^{(k)}, v^{(k)})\}_{k \in \mathbb{N}_0}$ in $\mathcal{M}_{\mathfrak{A}_n}^2(0, T; L^2(\mathcal{O})) \times \mathcal{M}_{\mathfrak{A}_n}^2(0, T; H^1(\mathcal{O}))$ by*

$$u_n^{(0)} \equiv 0, \quad (u_n^{(k)}, v^{(k)}) := (\mathfrak{T}_n(u_n^{(k-1)}), \mathfrak{R}(u_n^{(k-1)})), \quad k \geq 1. \quad (4.20)$$

Then the family $\{\text{Law}(u_n^{(k)}, v^{(k)}) : k \in \mathbb{N}\}$ is tight in $\mathcal{M}(C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O})))$.

Proof. Since tightness of a product family follows from tightness of each marginal, we treat the two components separately.

Tightness of $\{u_n^{(k)}\}$: By Lemma 4.17 and induction, $u_n^{(0)} = 0 \in \mathcal{C}_R$ and $u_n^{(k)} = \mathfrak{T}_n(u_n^{(k-1)}) \in \mathcal{C}_R$ for all $k \geq 1$. Hence every law $\text{Law}(u_n^{(k)})$ belongs to the family $\{\text{Law}(\mathfrak{T}_n \xi) \mid \xi \in \mathcal{C}_R\}$, which is tight in $\mathcal{M}(C^0([0, T]; L^2(\mathcal{O})))$ by Lemma 4.26.

Tightness of $\{v^{(k)}\}$: By formula (4.9),

$$v^{(k)}(t) = e^{t\Delta} v_0 - (\mathfrak{F}_\Delta \eta^{(k)})(t), \quad \eta^{(k)}(s) := u_n^{(k-1)}(s) v^{(k)}(s).$$

The initial value satisfies $v^{(k)}(0) = v_0 \in H^1(\mathcal{O})$ $\mathbb{P}_n$ -a.s., so $\{\text{Law}(v^{(k)}(0))\} = \{\text{Law}(v_0)\}$ is a singleton family of probability measures on the Polish space $H^1(\mathcal{O})$, hence trivially tight. For equicontinuity: since $u_n^{(k-1)} \in \mathcal{C}_R$ and Lemma 4.11 gives $\|v^{(k)}\|_{L^\infty(0, T; H^1)} \leq C(T, v_0)$ $\mathbb{P}_n$ -a.s., the Sobolev embedding $H^1(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ (Lemma 2.13 in $d = 1$) and Lemma 4.23 yield $\eta^{(k)} \in L^q(0, T; L^2(\mathcal{O}))$ for $q \in (2, 2 + \varepsilon_u)$ with

$$\begin{aligned} \mathbb{E}_n[\|\eta^{(k)}\|_{L^q(0, T; L^2)}] &\leq \mathbb{E}_n[\|u_n^{(k-1)}\|_{L^q(0, T; L^2)} \|v^{(k)}\|_{L^\infty(0, T; L^2)}] \\ &\leq \mathbb{E}_n[\|v^{(k)}\|_{L^\infty(0, T; L^2)}^2]^{\frac{1}{2}} \mathbb{E}_n[\|u_n^{(k-1)}\|_{L^q(0, T; L^2)}^2]^{\frac{1}{2}} \\ &\leq C_1(n, T) \mathbb{E}_n[\|v_0\|_{H_\mathcal{O}^1}] \sqrt{R} \leq C_2(n, T) \sqrt{R} \quad \text{uniformly in } k. \end{aligned}$$

Applying Lemma 4.19(i) with $\alpha = 1$, $\rho = 0$ and $\beta \in (0, \frac{1}{2} - \frac{1}{q})$,

$$\|\mathfrak{F}_\Delta \eta^{(k)}\|_{C^\beta([0, T]; H^1)} \leq C \|\eta^{(k)}\|_{L^q(0, T; L^2)} \quad \mathbb{P}_n\text{-a.s.}$$

uniformly in k . Together with the continuity of $t \mapsto e^{t\Delta} v_0$ in $H^1(\mathcal{O})$, Markov's inequality and Theorem 2.45 yield tightness of $\{\text{Law}(v^{(k)})\}$ in $\mathcal{M}(C^0([0, T]; H^1(\mathcal{O})))$. $\square$

Lemma 4.43 (Step (b): Skorokhod representation). *There exist a probability space $\tilde{\mathfrak{A}}_n = (\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{\mathbb{P}}_n)$, a subsequence (still denoted (k)), random variables $\{(\tilde{u}_n^{(k)}, \tilde{v}^{(k)}) : k \in \mathbb{N}\}$ and a limit $(\tilde{u}_n^*, \tilde{v}^*)$, all with values in $C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$ and defined over $\tilde{\mathfrak{A}}_n$, such that*

$$(i) \quad \text{Law}((\tilde{u}_n^{(k)}, \tilde{v}^{(k)})) = \text{Law}((u_n^{(k)}, v^{(k)})) \text{ for all } k \in \mathbb{N},$$

$$(ii) \quad (\tilde{u}_n^{(k)}, \tilde{v}^{(k)}) \rightarrow (\tilde{u}_n^*, \tilde{v}^*) \quad \tilde{\mathbb{P}}_n\text{-a.s. in } C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O})) \text{ as } k \rightarrow \infty.$$

Proof. By Lemma 4.42, the family $\{\text{Law}(u_n^{(k)}, v^{(k)}) : k \in \mathbb{N}\}$ is tight in the Polish space $C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$. By Prokhorov's theorem (Theorem 2.43), there exist a subsequence and a Borel probability measure ρ_n^* such that $\text{Law}(u_n^{(k)}, v^{(k)}) \rightharpoonup \rho_n^*$. The Skorokhod representation theorem (Theorem 2.46) then provides the probability space $\tilde{\mathfrak{A}}_n$ and the processes with properties (i) and (ii). $\square$

Lemma 4.44 (Step (c): Martingale characterisation). *For each $k \in \mathbb{N}$, the process*

$$\begin{aligned} \tilde{M}_n^{(k)}(t) &:= \tilde{u}_n^{(k)}(t) - u_0 \\ &+ \int_0^t \left[(-\Delta - \gamma I) \tilde{u}_n^{(k)}(s) \right. \\ &\quad \left. + \psi_n(h^1(\tilde{u}_n^{(k-1)}, s)) \psi_n(h^2(\tilde{v}^{(k)}, s)) \psi_n(h^3(\tilde{v}^{(k)}, s)) \text{div}(\tilde{u}_n^{(k-1)}(s) \nabla \tilde{v}^{(k)}(s)) \right] ds \end{aligned}$$

is a local martingale (Definition 2.31) over $\tilde{\mathfrak{A}}_n$ with respect to the filtration

$$\tilde{G}_{n,t}^{(k)} := \sigma(\tilde{u}_n^{(k)}(s), \tilde{v}^{(k)}(s), \tilde{u}_n^{(k-1)}(s) : s \leq t)$$

and with quadratic variation

$$\langle \tilde{M}_n^{(k)} \rangle_t = \int_0^t \sum_{j \in \mathbb{Z}} \left\| \tilde{u}_n^{(k)}(s) e_j^{(\delta_1)} \right\|_{L^2}^2 ds, \quad t \in [0, T].$$

Moreover, $\tilde{v}^{(k)}$ satisfies $\tilde{\mathbb{P}}_n$ -a.s. for all $t \in [0, T]$

$$\tilde{v}^{(k)}(t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \tilde{u}_n^{(k-1)}(s) \tilde{v}^{(k)}(s) ds. \quad (4.21)$$

Proof. By Lemma 4.43(i), $\text{Law}(\tilde{u}_n^{(k)}, \tilde{v}^{(k)}, \tilde{u}_n^{(k-1)}) = \text{Law}(u_n^{(k)}, v^{(k)}, u_n^{(k-1)})$. Since $u_n^{(k)} = \mathfrak{T}_n(u_n^{(k-1)})$ satisfies formula (4.8) over $\mathfrak{A}_n$, the process $M_n^{(k)}$ (defined analogously with $u_n^{(k)}$, $v^{(k)}$, $u_n^{(k-1)}$ on $\mathfrak{A}_n$) is a local martingale with the stated quadratic variation, by the Itô isometry and the Wiener expansion $\mathcal{W} = \sum_{j \in \mathbb{Z}} e_j^{(\delta_1)} \beta_j^{(1)}$. The equal-law transfer to $\tilde{\mathfrak{A}}_n$ preserves both the martingale property and the quadratic variation identity. Equation (4.21) follows analogously from (4.9) via the equal-law property. $\square$

Lemma 4.45 (Step (d): Convergence in L^r). *The following two statements hold.*

$$(i) \sup_{k \in \mathbb{N}} \tilde{\mathbb{E}}_n \left[\sup_{t \in [0, T]} \|\tilde{u}_n^{(k)}(t)\|_{L^2}^2 \right] \leq R.$$

$$(ii) \text{ For any } r \in [1, 2 + \varepsilon_u),$$

$$\lim_{k \rightarrow \infty} \tilde{\mathbb{E}}_n \left[\sup_{t \in [0, T]} \|\tilde{u}_n^{(k)}(t) - \tilde{u}_n^*(t)\|_{L^2}^r \right] = 0.$$

Proof. Part (i): Since $\text{Law}(\tilde{u}_n^{(k)}) = \text{Law}(u_n^{(k)})$ and $u_n^{(k)} \in \mathcal{C}_R$,

$$\tilde{\mathbb{E}}_n \left[\sup_t \|\tilde{u}_n^{(k)}\|_{L^2}^2 \right] = \mathbb{E}_n \left[\sup_t \|u_n^{(k)}\|_{L^2}^2 \right] \leq R.$$

Part (ii): By Lemma 4.23 and the equal-law property,

$$\sup_{k \in \mathbb{N}} \tilde{\mathbb{E}}_n \left[\sup_t \|\tilde{u}_n^{(k)}\|_{L^2}^{2+\varepsilon_u} \right] \leq C(n, \varepsilon_u, T) < \infty.$$

Hence the family $\{\sup_t \|\tilde{u}_n^{(k)} - \tilde{u}_n^*\|_{L^2}^r\}_{k \in \mathbb{N}}$ is uniformly integrable with respect to $\tilde{\mathbb{P}}_n$ for every $r < 2 + \varepsilon_u$. Combined with $\tilde{u}_n^{(k)} \rightarrow \tilde{u}_n^*$ $\tilde{\mathbb{P}}_n$ -a.s. in $C^0([0, T]; L^2(\mathcal{O}))$ (Lemma 4.43(ii)), the Vitali convergence theorem yields part (ii). $\square$

Lemma 4.46 (Step (e): Limit martingale and Wiener process). *The limit martingale $\tilde{M}_n^*(t) := \lim_{k \rightarrow \infty} \tilde{M}_n^{(k)}(t)$ exists in $L^1(\tilde{\mathbb{P}}_n)$ for every $t \in [0, T]$ and the following hold.*

- (i) $\tilde{M}_n^*$ is a continuous local martingale over $\tilde{\mathfrak{A}}_n$ with respect to the filtration $\tilde{G}_{n,t}^* := \sigma(\tilde{u}_n^*(s), \tilde{v}^*(s) : s \leq t) \vee \mathcal{N}$ and quadratic variation

$$\langle \tilde{M}_n^* \rangle_t = \int_0^t \sum_{j \in \mathbb{Z}} \left\| \tilde{u}_n^*(s) e_j^{(\delta_1)} \right\|_{L^2}^2 ds.$$

- (ii) The limit $\tilde{v}^*$ satisfies $\tilde{\mathbb{P}}_n$ -a.s.

$$\tilde{v}^*(t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \tilde{u}_n^*(s) \tilde{v}^*(s) ds. \quad (4.22)$$

- (iii) There exist an extension $\tilde{\tilde{\mathfrak{A}}}_n$ of $\tilde{\mathfrak{A}}_n$ and a cylindrical Wiener process $\tilde{W}$ on $\mathcal{H}$ defined over $\tilde{\tilde{\mathfrak{A}}}_n$ such that

$$\tilde{M}_n^*(t) = \int_0^t \tilde{u}_n^*(s) d\tilde{W}(s).$$

Proof. Existence of the limit and part (i): By Lemma 4.45(ii) with $r = 2$, $\tilde{u}_n^{(k)} \rightarrow \tilde{u}_n^*$ in $L^2(\tilde{\mathbb{P}}_n; C^0([0, T]; L^2(\mathcal{O})))$. Passing $k \rightarrow \infty$ in $\tilde{M}_n^{(k)}$ term by term:

- the linear term $(-\Delta - \gamma I) \tilde{u}_n^{(k)}$ converges in L^2 by part (ii) with $r = 2$;
- the nonlinear term $\operatorname{div}(\tilde{u}_n^{(k-1)} \nabla \tilde{v}^{(k)})$ converges in H^{-1} since in $d = 1$, using $H^1 \hookrightarrow L^\infty$,

$$\begin{aligned} & \left\| \operatorname{div}((\tilde{u}_n^{(k-1)} - \tilde{u}_n^*) \nabla \tilde{v}^{(k)}) \right\|_{H^{-1}} \\ & \leq \|\tilde{u}_n^{(k-1)} - \tilde{u}_n^*\|_{L^2} \|\tilde{v}^{(k)}\|_{H^1} \rightarrow 0 \quad \tilde{\mathbb{P}}_n\text{-a.s.}, \end{aligned}$$

and analogously for the term with $\tilde{v}^{(k)} - \tilde{v}^*$.

This defines $\tilde{M}_n^*$ and, by passing to the limit in the martingale characterisation from Lemma 4.44, shows it is a local martingale with the stated quadratic variation.

Part (ii): Passing $k \rightarrow \infty$ in (4.21): by Lemma 4.43(ii) and $H^1(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$, $\tilde{u}_n^{(k-1)} \tilde{v}^{(k)} \rightarrow \tilde{u}_n^* \tilde{v}^*$ in $C^0([0, T]; L^2(\mathcal{O}))$ $\tilde{\mathbb{P}}_n$ -a.s. Dominated convergence then yields (4.22).

Part (iii): By the martingale representation theorem (e.g. [DZ14]), applied to $\tilde{M}_n^*$ which is a continuous local martingale with quadratic variation of the required form, there exist an extension $\tilde{\tilde{\mathfrak{A}}}_n$ and a cylindrical Wiener process $\tilde{W}$ on $\mathcal{H}$ such that $\tilde{M}_n^*(t) = \int_0^t \tilde{u}_n^*(s) d\tilde{W}(s)$. $\square$

Lemma 4.47 (Step (f): Fixed point). *Over the extended probability space $\tilde{\mathfrak{A}}_n$, together with the Wiener process $\tilde{\mathcal{W}}$ from Lemma 4.46(iii), the pair $(\tilde{u}_n^*, \tilde{v}^*)$ is a martingale solution to the truncated system (4.6). In particular, $\tilde{\mathfrak{T}}_n(\tilde{u}_n^*) = \tilde{u}_n^*$ $\tilde{\mathbb{P}}_n$ -a.s.*

Proof. It suffices to show $\tilde{\mathbb{E}}_n[\sup_t \|\tilde{\mathfrak{T}}_n(\tilde{u}_n^*) - \tilde{u}_n^*\|_{L^2}^2] \leq \varepsilon$ for every $\varepsilon > 0$. Decompose

$$\tilde{\mathfrak{T}}_n(\tilde{u}_n^*) - \tilde{u}_n^* = \underbrace{\tilde{\mathfrak{T}}_n(\tilde{u}_n^*) - \tilde{\mathfrak{T}}_n(\tilde{u}_n^{(k-1)})}_{=:A_k} + \underbrace{\tilde{\mathfrak{T}}_n(\tilde{u}_n^{(k-1)}) - \tilde{u}_n^{(k)}}_{=:B_k} + \underbrace{\tilde{u}_n^{(k)} - \tilde{u}_n^*}_{=:C_k}.$$

Term C_k : By Lemma 4.45(ii) with $r = 2$, $\tilde{\mathbb{E}}_n[\sup_t \|C_k\|_{L^2}^2] \rightarrow 0$.

Term B_k : By the equal-law property Lemma 4.43(i) and the definition (4.20) of the Picard sequence, $\text{Law}(\tilde{u}_n^{(k)}) = \text{Law}(u_n^{(k)}) = \text{Law}(\mathfrak{T}_n(u_n^{(k-1)})) = \text{Law}(\tilde{\mathfrak{T}}_n(\tilde{u}_n^{(k-1)}))$, hence $B_k = 0$ $\tilde{\mathbb{P}}_n$ -a.s.

Term A_k : By Lemma 4.41 (Hölder continuity of $\mathfrak{T}_n$ with exponent $\frac{1}{4}$),

$$\tilde{\mathbb{E}}_n \left[\sup_t \|A_k\|_{L^2}^2 \right] \leq C \left(\tilde{\mathbb{E}}_n \left[\sup_t \|\tilde{u}_n^* - \tilde{u}_n^{(k-1)}\|_{L^2}^2 \right] \right)^{1/4} \xrightarrow{k \rightarrow \infty} 0$$

by Lemma 4.45(ii) with $r = 2$.

Combining all three terms yields $\tilde{\mathbb{E}}_n[\sup_t \|\tilde{\mathfrak{T}}_n(\tilde{u}_n^*) - \tilde{u}_n^*\|_{L^2}^2] \leq \varepsilon$ for all $\varepsilon > 0$, and therefore $\tilde{\mathfrak{T}}_n(\tilde{u}_n^*) = \tilde{u}_n^*$ $\tilde{\mathbb{P}}_n$ -a.s. Together with Lemma 4.46, $(\tilde{u}_n^*, \tilde{v}^*, \tilde{\mathcal{W}})$ is a martingale solution to (4.6). $\square$

Proposition 4.48. *For each $n \in \mathbb{N}$ and any initial condition (u_0, v_0) satisfying Assumption 4.5, there exists a probability space $\mathfrak{A}_n^* = (\Omega_n^*, \mathcal{F}_n^*, (\mathcal{F}_{n,t}^*)_{t \in [0,T]}, \mathbb{P}_n^*)$, a cylindrical Wiener process $\mathcal{W}_n^*$ on $\mathcal{H}$ over $\mathfrak{A}_n^*$, and a pair (u_n^*, v_n^*) that is a martingale solution to the truncated system (4.6) with $(u_n^*, v_n^*) \in C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$ $\mathbb{P}_n^*$ -a.s. Moreover, there exists $C = C(n) > 0$ such that*

$$\mathbb{E}_n^* \left[\sup_{t \in [0, T]} \|u_n^*(t)\|_{L^2(\mathcal{O})}^2 \right] + \mathbb{E}_n^* \left[\sup_{t \in [0, T]} \|\nabla v_n^*(t)\|_{L^2(\mathcal{O})}^2 \right] \leq C(n).$$

Proof. The existence follows from Lemmata 4.42–4.47: Lemma 4.42 constructs the Picard sequence and establishes tightness of the associated laws; Lemma 4.43 applies Prokhorov's theorem and the Skorokhod representation to extract an a.s.-convergent realisation; Lemma 4.44 identifies the martingale structure inherited by the realised sequence; Lemma 4.45 establishes L^r -convergence for $r < 2 + \varepsilon_u$ via Vitali's theorem; Lemma 4.46 passes to the limit, identifies the integral equation for v^* , and constructs the cylindrical Wiener process $\tilde{\mathcal{W}}$ via the martingale representation theorem; and Lemma 4.47 verifies the fixed-point identity, showing that the limit is a martingale solution. Setting $\mathfrak{A}_n^* := \tilde{\mathfrak{A}}_n$ and $\mathcal{W}_n^* := \tilde{\mathcal{W}}$ completes the construction. The moment bound for u_n^* follows from Lemma 4.45(i) by taking $k \rightarrow \infty$, and the bound for ∇v_n^* follows from Lemma 4.11. $\square$

4.2.3 Step II: Non-negativity of Solutions

Non-negativity of u_n and v_n is needed both for the biological interpretation of u_n as a cell density and v_n as a concentration, and technically, since the truncated system (4.7) was only shown to be well-posed without requiring $u, v \geq 0$, whereas the passage to the limit in Step IV (Section 4.2.5) uses non-negativity to control $u \log u$ from below and to justify the maximum-principle-type bound on v . For this we will work towards $\|u_n^-(t)\|_{L^2_{\mathcal{O}}}^2 = \|v_n^-(t)\|_{L^2_{\mathcal{O}}}^2 = 0$ $\mathbb{P}_n$ -a.s. for a.e. $t \in [0, T]$. The norms will be the limit of a function ϕ_δ we now introduce.

Definition 4.49. For $x \in \mathbb{R}$ we define

$$x^+ := \max\{x, 0\} = \begin{cases} x, & x > 0, \\ 0, & x \leq 0, \end{cases} \quad \text{and} \quad x^- := \max\{-x, 0\} = \begin{cases} -x, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

Also for $\delta \in (0, 1)$ we define

$$\begin{aligned} g_\delta(r) &:= \frac{r^2}{r + \delta}, & \text{for all } r \in (-\delta, \infty), \\ G_\delta(r) &:= g_\delta((r^-)^2), & \text{for all } r \in \mathbb{R}, \\ \phi_\delta(w) &:= \int_{\mathcal{O}} G_\delta(w(x)) \, dx, & \text{for all } w \in L^2(\mathcal{O}). \end{aligned}$$

We first obtain some properties about these functions which will then be used later in this section.

Lemma 4.50. For the functions from definition 4.49, $G_\delta \in C^2(\mathbb{R})$ and the following hold.

- (i) For $r \in \mathbb{R}$: $|G_\delta(r)| \leq (r^-)^2$, $|G'_\delta(r)| \leq 2r^-$, $|G''_\delta(r)| \leq 10$,
 $G_\delta(r) \geq 0$, $G'_\delta(r) \leq 0$, $G''_\delta(r) \geq 0$,
- (ii) for $r \geq 0$: $G_\delta(r) = G'_\delta(r) = G''_\delta(r) = 0$,
- (iii) for $\delta \rightarrow 0$: $g_\delta \rightarrow \text{id}$, $g'_\delta \rightarrow 1$, $g''_\delta \rightarrow 0$, $G_\delta(r) \rightarrow (r^-)^2$,
 $G'_\delta(r) \rightarrow -2r^-$, $G''_\delta(r) \rightarrow 2$, $\phi_\delta(w) \rightarrow \|w^-\|_{L^2_{\mathcal{O}}}^2$, $\forall r \in \mathbb{R} \forall w \in L^2(\mathcal{O})$.

Proof. (i) It suffices to prove this statement for $r < 0$ since $r \geq 0$ is handled in (ii). Since for all $r > 0$, $g_\delta(r) \leq r$, $|G_\delta(r)| \leq (r^-)^2$ is obvious. Easy calculations show for $r > 0$

$$g'_\delta(r) = \frac{r^2 + 2r\delta}{(r + \delta)^2} \leq 1 \quad \text{and} \quad g''_\delta(r) = \frac{2\delta^2}{(r + \delta)^3},$$

and hence for $r < 0$, $|G'_\delta(r)| = |\partial_r[g_\delta(r^2)]| = |2rg'_\delta(r^2)| \leq 2|r| = 2r^-$ as well as

$$G''_\delta(r) = \partial_r^2[g_\delta(r^2)] = 2g'_\delta(r^2) + 4r^2g''_\delta(r^2) \leq 2 + \frac{8r^2\delta^2}{(r^2 + \delta)^3} \leq 2 + 8 = 10.$$

(ii) For $r > 0$ the statement directly follows by $r^- = 0$, hence $r \searrow 0$ is discussed. Using

the formulas derived in (i) we compute the left-hand limits as $r \nearrow 0$:

$$\begin{aligned}\lim_{r \nearrow 0} G_\delta(r) &= \lim_{r \nearrow 0} g_\delta(r^2) = g_\delta(0) = 0, \\ \lim_{r \nearrow 0} G'_\delta(r) &= \lim_{r \nearrow 0} 2r g'_\delta(r^2) = 0, \\ \lim_{r \nearrow 0} G''_\delta(r) &= \lim_{r \nearrow 0} [2g'_\delta(r^2) + 4r^2 g''_\delta(r^2)] = 2g'_\delta(0) + 0 = 0,\end{aligned}$$

Since left- and right-hand limits coincide at $r = 0$, the functions G_δ , G'_δ , and G''_δ are all continuous on $\mathbb{R}$, hence $G_\delta \in C^2(\mathbb{R})$.

- (iii) The first two statements are obvious. For the third we notice $|G_\delta(w(x))| \leq (w(x)^-)^2 \leq |w(x)|^2$ for all $x \in \mathcal{O}$ by (ii). Since $w \in L^2(\mathcal{O})$ implies $w^2 \in L^1(\mathcal{O})$, applying Lebesgue's dominated convergence theorem shows the claim. $\square$

Next we want to apply the Itô formula to $\phi_\delta(u_n(t))$ but since Δ is not bounded from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ we will have to work with its Yosida Approximation. Hence we apply Y_ε to the mild solution u_n .

Lemma 4.51. *The Yosida approximation $u_n^\varepsilon = Y_\varepsilon u_n$ is the unique strong solution to*

$$du_n^\varepsilon(t) = \left(Bu_n^\varepsilon(t) - \Psi_n(t) Y_\varepsilon [\operatorname{div}(u_n \nabla v_n)(t)] \right) dt + u_n^\varepsilon dW(t), \quad (4.23)$$

where $\Psi_n := \psi_n(h^1(u_n, \cdot)) \psi_n(h^2(v_n, \cdot)) \psi_n(h^3(v_n, \cdot))$, i.e., for all $t \in [0, T]$

$$u_n^\varepsilon(t) = u_0^\varepsilon + \int_0^t (Bu_n^\varepsilon(s) - \Psi_n(s) Y_\varepsilon \operatorname{div}(u_n \nabla v_n)(s)) ds + \int_0^t u_n^\varepsilon(s) dW(s) \quad \mathbb{P}_n\text{-a.s.}$$

Proof. Since u_n is a mild solution to $du_n = (Bu_n - \Psi_n \operatorname{div}(u_n \nabla v_n)) dt + u_n dW(t)$, i.e.,

$$u_n(t) = e^{tB} u_0 - \int_0^t e^{(t-s)B} \Psi_n(s) \operatorname{div}(u_n(s) \nabla v_n(s)) ds + \int_0^t e^{(t-s)B} u_n(s) dW(s) \quad \mathbb{P}_n\text{-a.s.},$$

which after application of Y_ε leads to

$$u_n^\varepsilon(t) = e^{tB} u_0^\varepsilon - \int_0^t e^{(t-s)B} \Psi_n(s) Y_\varepsilon \operatorname{div}(u_n(s) \nabla v_n(s)) ds + \int_0^t e^{(t-s)B} u_n^\varepsilon(s) dW(s) \quad \mathbb{P}_n\text{-a.s.},$$

because Y_ε and $e^{\cdot B}$ commute. Hence u_n^ε is a mild solution to (4.23). Furthermore because

$$\mathbb{E}_n \left[\int_0^T \|G(u_n^\varepsilon(t))\|_{\text{HS}}^2 dt \right] \leq \gamma \mathbb{E}_n \left[\int_0^T \|Y_\varepsilon u_n\|_{L^2}^2 dt \right] \leq \gamma \mathbb{E}_n \left[\sup_{t \in [0, T]} \|u_n(t)\|_{L^2}^2 \right] < \infty,$$

by the equivalence of mild and weak solutions for semigroup-generated equations (see e.g. [Paz83, Chapter 4]) u_n^ε also is a weak solution to (4.23) and since $u_n^\varepsilon \in \operatorname{dom}(B)$ it is also a strong solution to (4.23). By uniqueness of mild solutions to this linear equation ([Paz83, Chapter 4]) the solution is also unique and the representation for u_n^ε follows by the definition of strong solutions. $\square$

Now we can apply the Itô formula (Theorem 2.37) to ϕ_δ on $u_n^\varepsilon(t)$. Doing so we get

$$\phi_\delta(u_n^\varepsilon(t)) = \phi_\delta(u_0^\varepsilon) + \int_0^t \sum_{j=1}^4 I_j^{\varepsilon,\delta}(s) \, ds + \underbrace{\int_0^t I_5^{\varepsilon,\delta}(s) \, dW(s)}_{=: M_\delta^\varepsilon(t)}$$

with

$$\begin{aligned} I_1^{\varepsilon,\delta}(s) &= \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) \Delta u_n^\varepsilon(s, x) \, dx, \\ I_2^{\varepsilon,\delta}(s) &= \gamma \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) u_n^\varepsilon(s, x) \, dx, \\ I_3^{\varepsilon,\delta}(s) &= -\Psi_n(s) \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) Y_\varepsilon \operatorname{div}(u_n \nabla v_n)(s, x) \, dx, \\ I_4^{\varepsilon,\delta}(s) &= \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_{\mathcal{O}} G''_\delta(u_n^\varepsilon(s, x)) (u_n^\varepsilon(s, x) e_k^{(\delta_1)}(x))^2 \, dx, \\ I_5^{\varepsilon,\delta}(s) &= \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) u_n^\varepsilon(s, x) \, dx. \end{aligned}$$

Lemma 4.52. *For ϕ_δ , the following holds $\mathbb{P}_n$ -a.s. for all $t \in [0, T]$*

$$\begin{aligned} \phi_\delta(u_n^\varepsilon(t)) &\leq \phi_\delta(u_0^\varepsilon) + (2\gamma + 5\gamma_1) \int_0^t \|(u_n^\varepsilon)^-(s)\|_{L^2}^2 \, ds \\ &\quad + 8C_E^2 \int_0^t \|v_n(s)\|_{H^2}^2 \left(\|(u_n^\varepsilon)^-(s)\|_{L^2}^2 + \|u_n(s) - u_n^\varepsilon(s)\|_{L^2}^2 \right) \, ds + M_\delta^\varepsilon(t). \end{aligned}$$

Proof. Leaving out (s, x) in the calculations we see that for $I_1^{\varepsilon,\delta}$

$$I_1^{\varepsilon,\delta} = - \int_{\mathcal{O}} G''_\delta(u_n^\varepsilon) |\nabla u_n^\varepsilon|^2 \leq -\frac{1}{2} \int_{\mathcal{O}} G''_\delta(u_n^\varepsilon) |\nabla u_n^\varepsilon|^2.$$

For $I_2^{\varepsilon,\delta}$ we use $|G'_\delta(r)| \leq 2|r| \cdot \mathbf{1}_{\{r < 0\}}$ from Lemma 4.50 for

$$I_2^{\varepsilon,\delta} \leq 2\gamma \int_{\mathcal{O}} |(u_n^\varepsilon)^-|^2 = 2\gamma \|(u_n^\varepsilon)^-\|_{L^2_{\mathcal{O}}}^2.$$

For $I_4^{\varepsilon,\delta}$ we obtain something similar by $|G''_\delta(r)| \leq 10$ from Lemma 4.50(i)

$$I_4^{\varepsilon,\delta} \leq 5 \int_{\mathcal{O}} \sum_{k \in \mathbb{Z}} \left((u_n^\varepsilon)^- e_k^{(\delta_1)} \right)^2 = 5\gamma_1 \int_{\mathcal{O}} |(u_n^\varepsilon)^-|^2 = 5\gamma_1 \|(u_n^\varepsilon)^-\|_{L^2_{\mathcal{O}}}^2.$$

For $I_3^{\varepsilon,\delta}$ we use, that Y_ε is self-adjoint, commutes with $(-\Delta_N)^{\frac{1}{2}}$ on H_N^1 which in norms is the same as ∇ and Y_ε is a contraction, the embedding $H^1 \hookrightarrow L^\infty$ (with constant C_E) in $d = 1$

and Young

$$\begin{aligned}
I_3^{\varepsilon, \delta} &= -\Psi_n \int_{\mathcal{O}} Y_\varepsilon [G'_\delta(u_n^\varepsilon)] \operatorname{div}(u_n \nabla v_n) = \Psi_n \int_{\mathcal{O}} \nabla [Y_\varepsilon G'_\delta(u_n^\varepsilon)] u_n \nabla v_n \\
&\leq \|(-\Delta)^{\frac{1}{2}} Y_\varepsilon G'_\delta(u_n^\varepsilon)\|_{L^2} \|(u_n^\varepsilon)^- + u_n - u_n^\varepsilon\|_{L^2} \|\nabla v_n\|_{L^\infty} \\
&\leq \|(-\Delta)^{\frac{1}{2}} G'_\delta(u_n^\varepsilon)\|_{L^2} \|(u_n^\varepsilon)^- + u_n - u_n^\varepsilon\|_{L^2} C_E \|v_n\|_{H^2} \\
&\leq \|G''_\delta(u_n^\varepsilon) \nabla u_n^\varepsilon\|_{L^2} \|(u_n^\varepsilon)^- + u_n - u_n^\varepsilon\|_{L^2} C_E \|v_n\|_{H^2} \\
&\leq \left\| \sqrt{G''_\delta(u_n^\varepsilon)} \nabla u_n^\varepsilon \right\|_{L^2} \cdot \sqrt{8} C_E \|(u_n^\varepsilon)^- + u_n - u_n^\varepsilon\|_{L^2} \|v_n\|_{H^2} \\
&\leq \frac{1}{2} \int_{\mathcal{O}} G''_\delta(u_n^\varepsilon) |\nabla u_n^\varepsilon|^2 + \frac{\sqrt{8}^2 \cdot 2C_E^2}{2} (\|(u_n^\varepsilon)^-\|_{L^2}^2 + \|u_n - u_n^\varepsilon\|_{L^2}^2) \|v_n\|_{H^2}^2,
\end{aligned}$$

where the first addend is absorbed in $I_1^{\varepsilon, \delta}$. $\square$

Lemma 4.53. *As $\varepsilon, \delta \rightarrow 0$, $M_\delta^\varepsilon(t)$ converges to $M(t)$ in probability, where*

$$M(t) := 2 \int_0^t \int_{\mathcal{O}} u_n^-(s, x)^2 \, dx \, dW(s), \quad \text{i.e.,} \quad \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathbb{P}_n(|M_\delta^\varepsilon(t) - M(t)| > \varepsilon_0) = 0 \quad \forall \varepsilon_0 > 0.$$

Proof. Because of the Markov inequality it is sufficient to show

$$\mathbb{E}_n[|M_\delta^\varepsilon(t) - M(t)|^2] = \mathbb{E}_n \left[\int_0^t \left| \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) u_n^\varepsilon(s, x) - u_n^-(s, x)^2 \, dx \right|^2 \, ds \right] \rightarrow 0$$

because of the Itô isometry (see e.g. [DZ14, Section 4.2]). Since $u_n^\varepsilon \rightarrow u_n$ in $L^2(\mathcal{O})$, $u_n^{\varepsilon_k} \rightarrow u_n$ pointwise a.e. in $\mathcal{O}$ for a subsequence $(\varepsilon_k)_k$ and because $G_\delta \in C^2(\mathbb{R}; \mathbb{R})$ we have $G'_\delta(u_n^{\varepsilon_k}) u_n^{\varepsilon_k} \rightarrow G'_\delta(u_n) u_n$ a.e. in $\mathcal{O}$. Because $|G'_\delta(u_n^\varepsilon) u_n^\varepsilon| \leq 2|u_n^\varepsilon|^2 \in L^1(\mathcal{O})$ $\mathbb{P}_n$ -a.s. in $[0, T]$, we also have by Lebesgue's dominated convergence theorem (DCT)

$$\int_{\mathcal{O}} G'_\delta(u_n^{\varepsilon_k}) u_n^{\varepsilon_k} \xrightarrow{\varepsilon_k \rightarrow 0} \int_{\mathcal{O}} G'_\delta(u_n) u_n \xrightarrow{\delta \rightarrow 0} 2 \int_{\mathcal{O}} (u_n^-)^2.$$

Using

$$\int_0^t \left| \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) u_n^\varepsilon(s, x) - u_n^-(s, x)^2 \, dx \right|^2 \leq 4 \int_0^t \|u_n^-(s)\|_{L^2_{\mathcal{O}}}^4 \, ds \leq 4 \sup_{s \in [0, t]} \|u_n^-(s)\|_{L^2_{\mathcal{O}}}^4 \in L^1(\Omega),$$

which implies

$$\left| \int_{\mathcal{O}} G'_\delta(u_n^\varepsilon(s, x)) u_n^\varepsilon(s, x) - u_n^-(s, x)^2 \, dx \right|^2 \leq 4 \|u_n^-(s)\|_{L^2_{\mathcal{O}}}^4 \in L^\infty(0, T) \hookrightarrow L^1(0, T) \quad \mathbb{P}_n\text{-a.s.},$$

the DCT implies the statement. $\square$

Corollary 4.54. *In the setting of Lemma 4.52 for the mild solution u_n fulfilling $u_0 \geq 0$ $\mathbb{P}_n$ -a.s. the following holds $\mathbb{P}_n$ -a.s. for all $t \in [0, T]$*

$$\|u_n^-(t)\|_{L^2_{\mathcal{O}}}^2 \leq (2\gamma + 5\gamma_1) \int_0^t \|u_n^-(s)\|_{L^2}^2 \, ds + 8C_E^2 \int_0^t \|v_n(s)\|_{H^2}^2 \|u_n^-(s)\|_{L^2}^2 \, ds + M(t).$$

Proof. We obtain that $|G_\delta(u_n^\varepsilon)| \leq |u_n^\varepsilon|^2 \in L^1(\mathcal{O})$ $\mathbb{P}_n$ -a.s. in $[0, T]$ because

$$\int_{\mathcal{O}} |u_n^\varepsilon|^2 = \|u_n^\varepsilon\|_{L^2_{\mathcal{O}}}^2 \leq \|u_n\|_{L^2_{\mathcal{O}}}^2 < \infty \quad \mathbb{P}_n\text{-a.s. in } [0, T].$$

Hence by Lemma 4.50(iii) and Lebesgue's dominated convergence theorem (DCT) on $\mathcal{O}$ we have $\phi_\delta(u_n^\varepsilon) \rightarrow \|(u_n^\varepsilon)^-\|_{L^2_{\mathcal{O}}}^2$ $\mathbb{P}_n$ -a.s. in $[0, T]$ as $\delta \rightarrow 0$.

Since $u_n^\varepsilon \rightarrow u_n$ in $L^2(\mathcal{O})$ there exists a subsequence $(\varepsilon_k)_k$ such that $u_n^{\varepsilon_k}(x) \rightarrow u_n(x)$ for a.e. $x \in \mathcal{O}$ and we have $|(u_n^\varepsilon)^-|^2 - |(u_n)^-|^2 \leq |u_n^\varepsilon|^2 + |u_n|^2 \in L^1(\mathcal{O})$ $\mathbb{P}_n$ -a.s. in $[0, T]$ because

$$\int_{\mathcal{O}} (|u_n^\varepsilon|^2 + |u_n|^2) = \|u_n^\varepsilon\|_{L^2_{\mathcal{O}}}^2 + \|u_n\|_{L^2_{\mathcal{O}}}^2 \leq 2\|u_n\|_{L^2_{\mathcal{O}}}^2 < \infty \quad \mathbb{P}_n\text{-a.s. in } [0, T].$$

Hence by DCT on $\mathcal{O}$ $\|(u_n^\varepsilon)^-\|_{L^2_{\mathcal{O}}}^2 \rightarrow \|u_n^-\|_{L^2_{\mathcal{O}}}^2$ $\mathbb{P}_n$ -a.s. in $[0, T]$ as $\varepsilon \rightarrow 0$. The limits of the deterministic integrals on the right hand side are obtained by DCT on $[0, T]$ with $\|u_n\|_{L^2_{\mathcal{O}}}^2 \in L^\infty(0, T) \hookrightarrow L^1(0, T)$.

Since $\mathbb{P}_n$ -a.s. convergence implies convergence in probability, which we have shown for M_δ^ε , we can apply convergence in probability on everything.

Finally $\|u_0^-\|_{L^2_{\mathcal{O}}}^2$ disappears $\mathbb{P}_n$ -a.s. because $u_0 \geq 0$ $\mathbb{P}_n$ -a.s. $\square$

Proposition 4.55. *Under Assumption 4.5, $u_n \geq 0$ holds a.e. in $[0, T] \times \mathcal{O} \times \Omega$.*

Proof. Let $\varphi(t) := \|u_n^-(t)\|_{L^2_{\mathcal{O}}}^2$, $g(t) := \|v_n(t)\|_{H^2}^2$ and $\kappa(t) := C(g(t) + 1)$ where we set $C = \max\{2\gamma + 5\gamma_1, 8C_E^2\}$. Then Corollary 4.54 implies

$$\varphi(t) \leq N(t) := \int_0^t \varphi(s)\kappa(s) ds + M(t), \quad \varphi(0) = N(0) = 0,$$

with $N \geq 0$ $\mathbb{P}_n$ -a.s. in $[0, T]$. Define $\Lambda(t) := \exp(-\int_0^t \kappa(s) ds) \in (0, 1]$ which fulfills $d\Lambda(t) = -\kappa(t)\Lambda(t) dt$. Define $Z(t) = \Lambda(t)N(t)$ which by the product rule fulfills

$$dZ(t) = \Lambda(t)(\varphi(t) - N(t))\kappa(t) dt + \Lambda(t) dM(t), \quad Z(0) = 0.$$

Since $\Lambda, N \geq 0$ we have

$$0 \leq Z(t) = \int_0^t \Lambda(s)(\varphi(s) - N(s))\kappa(s) ds + \int_0^t \Lambda(s) dM(s) =: A(t) + \tilde{M}(t),$$

with $A \leq 0$ because $\varphi \leq N$. Hence $\tilde{M} = Z - A \geq 0$ is a local, non-negative martingale, thus a supermartingale by Fatou's lemma, with

$$\mathbb{E}_n[\tilde{M}(t)] \leq \tilde{M}(0) = 0 \implies \tilde{M} = 0 \quad \mathbb{P}_n\text{-a.s. in } [0, T].$$

Then we have $\mathbb{P}_n$ -a.s. (because $Z \geq 0$ and $\Lambda > 0$)

$$Z = A \leq 0 \implies \Lambda N = Z = 0 \implies N = 0 \implies \varphi \leq N = 0 \implies \varphi = 0 \quad \text{in } [0, T],$$

which yields the claim. $\square$

Now we can also prove non-negativity of v_n , which is way easier to show because we already

have $v_n \in L^2(0, T; H^2(\mathcal{O}))$ which implies $v_n(t) \in H^2 = \text{dom}(-\Delta_N)$ for a.e. $t \in [0, T]$. Hence we do not need any approximation.

Proposition 4.56. *Under Assumption 4.5, $v_n \geq 0$ holds a.e. in $[0, T] \times \mathcal{O} \times \Omega$.*

Proof. By standard linear semigroup theory ([Paz83, Chapter 4]) there exists a unique strong solution $v_n \in C^0([0, T]; H^1(\mathcal{O})) \cap L^2(0, T; H^2(\mathcal{O}))$, i.e.,

$$v_n(t) - v_0 = \int_0^t \Delta v_n - \int_0^t u_n v_n \quad \mathbb{P}_n\text{-a.s. for } t \in [0, T],$$

which after application of the chain rule and $\phi_\delta(v_0) = 0$ is just

$$\begin{aligned} 0 \leq \phi_\delta(v_n(t)) &= \int_0^t \partial_s \phi_\delta(v_n(s)) \, ds = \int_0^t \int_{\mathcal{O}} G'_\delta(v_n) (\Delta v_n - u_n v_n) \\ &= - \int_0^t \int_{\mathcal{O}} \underbrace{G''_\delta(v_n)}_{\geq 0} |\nabla v_n|^2 - \int_0^t \int_{\mathcal{O} \cap \{v_n \leq 0\}} \underbrace{G'_\delta(v_n) u_n}_{\leq 0} \underbrace{v_n}_{\leq 0} \leq 0 \quad \mathbb{P}_n\text{-a.s. for } t \in [0, T], \end{aligned}$$

because $G'_\delta(r) = 0$ for $r \geq 0$. Taking $\delta \rightarrow 0$ by DCT ($|G_\delta(v_n)| \leq |v_n|^2 \in L^1(\mathcal{O})$) we get $\|v_n^-(t)\|_{L^2_{\mathcal{O}}}^2 = 0$ $\mathbb{P}_n$ -a.s. for all $t \in [0, T]$, which yields the claim. $\square$

The result slightly differs from the result of [HMT22a], who have shown $\mathbb{E}_n [\|u_n^-(t)\|_{L^2}^2] = 0$ for all $t \in [0, T]$.¹

4.2.4 Step III: Construction of a Sequence of Solutions up to Stopping Times

In contrast to [HMT22a], we will not construct $(\bar{u}_1, \bar{v}_1)$ and $(\bar{u}_2, \bar{v}_2)$ but we will construct $(\bar{u}_1, \bar{v}_1)$ and then inductively construct $(\bar{u}_{n+1}, \bar{v}_{n+1})$ on top of $(\bar{u}_n, \bar{v}_n)$. The solution pair $(\bar{u}_n, \bar{v}_n)$ will be a solution of the original system up to the timepoint $\bar{\tau}_n$, and from there on it will follow a linear system we now first examine.

Definition 4.57. For fixed $\sigma \in [0, T]$, $\bar{\mathcal{W}}$ cylindrical wiener process and $(y_0, z_0) \in L^2(\mathcal{O}) \times H^1(\mathcal{O})$ we define the linear system on $[0, T - \sigma] \times \mathcal{O}$

$$\begin{cases} dy = (\Delta y + \gamma y) \, dt + y \, d(\theta_\sigma \bar{\mathcal{W}})(t), \\ \partial_t z = \Delta z, \\ (y(0), z(0)) = (y_0, z_0), \end{cases} \quad (4.24)$$

with Neumann boundary conditions $(y(t), z(t)) \in D(A)$, $D(A) = \{u \in H^2(\mathcal{O}) \mid \partial_\nu u|_{\partial\mathcal{O}} = 0\}$, cf. [HMT22a, (2.1)], where θ_σ is the shift operator, mapping $f(t)$ to $f(t + \sigma)$.

Set $\tilde{\mathcal{W}}(t) := \bar{\mathcal{W}}(t + \sigma) - \bar{\mathcal{W}}(\sigma)$ for $t \in [0, T - \sigma]$.

¹I have not found a way to prove this result after the following inequality is obtained:

$$\mathbb{E} [\|u_n^-(t)\|_{L^2}^2] \leq C(r_u, \chi) \mathbb{E} \left[\int_0^t \int_{\mathcal{O}} |u_n^-(s, x)|^2 |\nabla v_n(s, x)|^2 \, dx \, ds \right] + (2\gamma + 4) \mathbb{E} \left[\int_0^t \|u_n^-(s)\|_{L^2}^2 \, ds \right]$$

Lemma 4.58. $\tilde{W}$ is a cylindrical Wiener process on $\mathcal{H}$ (not on two separate channels $\mathcal{H}_1, \mathcal{H}_2$ as in [HMT22a], since (4.1) has only one noise term), independent of $\mathcal{F}_\sigma$, adapted to $\mathcal{F}_{\sigma+t}$ and fulfills for all $t \in [0, T - \sigma]$,

$$d\tilde{W}(t) = d(\theta_\sigma \bar{W})(t).$$

Proof. Write $\bar{W} = \sum_k e_k^{(\delta_1)} \beta_k$ with independent real $(\mathcal{F}_t)$ -Brownian motions β_k . For deterministic c , the defining property of $(\mathcal{F}_t)$ -Brownian motion (conditional characteristic function of increments equals the unconditional one) gives $(\beta_k(c + \cdot) - \beta_k(c))_k \perp \mathcal{F}_c$, jointly over all k . Approximate σ by dyadic stopping times $\sigma_m \downarrow \sigma$ taking finitely many values, apply the deterministic statement on each atom $\{\sigma_m = jT/2^m\}$, and let $m \rightarrow \infty$: right-continuity of $(\mathcal{F}_t)$ gives $\mathcal{F}_\sigma = \bigcap_m \mathcal{F}_{\sigma_m}$, path-continuity of β_k gives a.s. convergence of the increments, and dominated convergence transfers the factorization to σ . For the identity we just calculate

$$d(\theta_\sigma \bar{W})(t) = d[\bar{W}(t + \sigma)] = d[\tilde{W}(t) + \bar{W}(\sigma)] = d\tilde{W}(t). \quad \square$$

Lemma 4.59. *There exists a unique (up to indistinguishability) mild solution pair (y, z) to system (4.24).*

Proof. The equation for z is the deterministic linear heat equation with Neumann boundary conditions, whose unique mild solution $z(t) = e^{t\Delta} z_0$ is classical semigroup theory (Lemma 2.19). For y , the coefficients $A = \Delta + \gamma I$ (bounded perturbation of a semigroup generator) and $G(y)h := yh$ (as in Section 4.2.1, a fixed bounded linear map $L^2(\mathcal{O}) \rightarrow L_2(\mathcal{H}, L^2(\mathcal{O}))$) satisfy the hypotheses of Theorem 2.33 (with $f \equiv 0$), applied on the shifted, independent noise $\theta_\sigma \bar{W}$. $\square$

Lemma 4.60. *The solutions y and z from before are non-negative.*

Proof. The non-negativity of y can be proven similar (even easier) to the of u_n and $y \geq 0$ follows by Hopf Maximum Principle for Neumann boundary condition. $\square$

Without proof we state the following lemma as a transfer help in the induction step.

Lemma 4.61. *Let $\mathfrak{A}_1 = (\Omega_1, \mathcal{F}_1, (\mathcal{F}_{1,t})_{t \in [0, T]}, \mathbb{P}_1)$ be a complete filtered probability space satisfying the usual conditions, let σ be an $(\mathcal{F}_{1,t})$ -stopping time with $\sigma \leq T$ $\mathbb{P}_1$ -a.s., and let $(\bar{u}, \bar{v}) \in C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$ $\mathbb{P}_1$ -a.s. be $(\mathcal{F}_{1,t})$ -adapted. Set $E := L^2(\mathcal{O}) \times H^1(\mathcal{O})$, $\eta := (\bar{u}(\sigma), \bar{v}(\sigma))$, and suppose $\mu := \text{Law}_{\mathbb{P}_1}(\eta)$ satisfies Assumption 4.5.*

Let $(\mathfrak{A}_2, \mathcal{W}_2, (u_2, v_2))$, $\mathfrak{A}_2 = (\Omega_2, \mathcal{F}_2, (\mathcal{F}_{2,t})_{t \in [0, T]}, \mathbb{P}_2)$, be any martingale solution to (4.6) (at the truncation level under consideration) with $(u_2(0), v_2(0))$ of law μ .

Then there exist a complete filtered probability space $\hat{\mathfrak{A}} = (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \in [0, T]}, \hat{\mathbb{P}})$ with

$$\hat{\Omega} = \Omega_1 \times X_2, \quad X_2 := C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O})) \times C([0, T]; \mathbb{R})^{\mathbb{Z}},$$

a canonical projection $\pi_1: \hat{\Omega} \rightarrow \Omega_1$ with $(\pi_1)_* \hat{\mathbb{P}} = \mathbb{P}_1$ ($f_* \mu$ meaning the pushforward measure $f_* \mu(A) := \mu(f^{-1}(A))$), a cylindrical Wiener process $\hat{\mathcal{W}}_2$ on $\mathcal{H}$ over $\hat{\mathfrak{A}}$, and a pair $(\hat{u}_2, \hat{v}_2)$, such that $(\hat{\mathfrak{A}}, \hat{\mathcal{W}}_2, (\hat{u}_2, \hat{v}_2))$ is a martingale solution to (4.6), and $\hat{\mathbb{P}}$ -a.s.

$$(i) \quad (\hat{u}_2(0), \hat{v}_2(0)) = \eta \circ \pi_1,$$

(ii) $\hat{\mathcal{W}}_2$ is independent of $\pi_1^{-1}(\mathcal{F}_1)$; in particular, of $\pi_1^{-1}(\mathcal{F}_{1,\sigma})$.

Proposition 4.62. *For every $n \in \mathbb{N}$ there exist a complete filtered probability space $\bar{\mathfrak{A}}_n = (\bar{\Omega}_n, \bar{\mathcal{F}}_n, (\bar{\mathcal{F}}_{n,t})_{t \in [0,T]}, \bar{\mathbb{P}}_n)$ satisfying the usual conditions, a cylindrical Wiener process $\bar{\mathcal{W}}_n$ on $\mathcal{H}$ over $\bar{\mathfrak{A}}_n$, a pair $(\bar{u}_n, \bar{v}_n) \in C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$ $\bar{\mathbb{P}}_n$ -a.s., and a $(\bar{\mathcal{F}}_{n,t})$ -stopping time $\bar{\tau}_n \leq T$, such that $(\bar{\mathfrak{A}}_n, \bar{\mathcal{W}}_n, (\bar{u}_n, \bar{v}_n))$ is a martingale solution to the truncated system (4.6) (at level n) up to $\bar{\tau}_n$, and on $[\bar{\tau}_n, T]$ the pair $(\bar{u}_n, \bar{v}_n)$ solves the linear system (4.24) with $\sigma = \bar{\tau}_n$, $\bar{\mathcal{W}} = \bar{\mathcal{W}}_n$ and initial data $(\bar{u}_n(\bar{\tau}_n), \bar{v}_n(\bar{\tau}_n))$.*

Proof. The proof is carried out as a mathematical induction.

Induction base: For $n = 1$ by Proposition 4.48 there exists a probability space $\mathfrak{A}_1 = (\Omega_1, \mathcal{F}_1, (\mathcal{F}_{1,t})_{t \in [0,T]}, \mathbb{P}_1)$, a cylindrical Wiener process $\mathcal{W}_1$ on $\mathcal{H}$ over $\mathfrak{A}_1$, and a pair (u_1, v_1) that is a martingale solution to the truncated system (4.6) (with $n = 1$) with $(u_1, v_1) \in C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$ $\mathbb{P}_1$ -a.s.

Now set $\tau_1^* := \min(\tau_1^1, \tau_1^2, \tau_1^3, T)$ where

$$\begin{aligned} \tau_1^1 &:= \inf \{s \geq 0 \mid \|u_1(s)\|_{L^1_{\mathcal{O}}} \geq 1\}, \\ \tau_1^2 &:= \inf \{s \geq 0 \mid \|\nabla v_1(s)\|_{L^2_{\mathcal{O}}}^2 \geq 1\}, \\ \tau_1^3 &:= \inf \{s \geq 0 \mid \int_0^s \|\nabla v_1(r)\|_{H^1_{\mathcal{O}}}^2 dr \geq 1\}. \end{aligned}$$

Here we obtain that for $t \leq \tau_1^*$ the cutoff fulfills $\Psi_1(t) = 1$ and hence until τ_1^* the pair (u_1, v_1) is a solution to the original problem. Set $\bar{\mathfrak{A}}_1 := \mathfrak{A}_1$, $\bar{\mathcal{W}}_1 := \mathcal{W}_1$ and

$$(\bar{u}_1(t), \bar{v}_1(t)) := \begin{cases} (u_1(t), v_1(t)), & \text{for } t \in [0, \tau_1^*), \\ (y_1(t - \tau_1^*), z_1(t - \tau_1^*)), & \text{for } t \in [\tau_1^*, T]. \end{cases}$$

where y_1 and z_1 are the solutions to System 4.24 for $\sigma = \tau_1^*$, $\bar{\mathcal{W}} = \mathcal{W}_1$ and $(y_0, z_0) = (u_1(\tau_1^*), v_1(\tau_1^*))$. Then $\bar{u}_1$ and $\bar{v}_1$ are continuous because all parts are and the initial conditions for y and z are chosen exactly right.

Induction hypothesis: Assume there exists a $n \in \mathbb{N}$ such that for every $k \in [1, n] \cap \mathbb{N}$ there exists a probability space $\bar{\mathfrak{A}}_k = (\bar{\Omega}_k, \bar{\mathcal{F}}_k, (\bar{\mathcal{F}}_{k,t})_{t \in [0,T]}, \bar{\mathbb{P}}_k)$, a cylindrical Wiener process $\bar{\mathcal{W}}_k$ on $\mathcal{H}$ over $\bar{\mathfrak{A}}_k$, and a pair

$$(\bar{u}_k, \bar{v}_k) = \begin{cases} (u_1(t), v_1(t)), & \text{for } t \in [0, \bar{\tau}_1), \\ (u_2(t - \bar{\tau}_1), v_2(t - \bar{\tau}_1)), & \text{for } t \in [\bar{\tau}_1, \bar{\tau}_2), \\ \vdots \\ (u_k(t - \bar{\tau}_{k-1}), v_k(t - \bar{\tau}_{k-1})), & \text{for } t \in [\bar{\tau}_{k-1}, \bar{\tau}_k), \\ (y_k(t - \bar{\tau}_k), z_k(t - \bar{\tau}_k)), & \text{for } t \in [\bar{\tau}_k, T], \end{cases}$$

where y_k and z_k are the solutions to System 4.24 for $\sigma = \bar{\tau}_k$, $\bar{\mathcal{W}} = \bar{\mathcal{W}}_k$ and $(y_0, z_0) =$

$(u_k(\tau_k^*), v_k(\tau_k^*))$ and where $\bar{\tau}_k = \tau_1^* + \dots + \tau_k^*$, $\tau_k^* := \min(\tau_k^1, \tau_k^2, \tau_k^3, T)$ and

$$\begin{aligned}\tau_k^1 &:= \inf \{s \geq 0 \mid \|u_k(s)\|_{L^1_{\mathcal{O}}} \geq k\}, \\ \tau_k^2 &:= \inf \{s \geq 0 \mid \|\nabla v_k(s)\|_{L^2_{\mathcal{O}}}^2 \geq k\}, \\ \tau_k^3 &:= \inf \{s \geq 0 \mid \int_0^s \|\nabla v_k(r)\|_{H^1_{\mathcal{O}}}^2 dr \geq k\}.\end{aligned}$$

Furthermore assume that this is a martingale solution to the truncated system (4.6) up to $\bar{\tau}_k$ with $(\bar{u}_k, \bar{v}_k) \in C^0([0, T]; L^2(\mathcal{O})) \times C^0([0, T]; H^1(\mathcal{O}))$ $\bar{\mathbb{P}}_k$ -a.s.

Induction step: Suppose $(\bar{\mathfrak{A}}_n, \bar{\mathcal{W}}_n, (\bar{u}_n, \bar{v}_n))$ and $\bar{\tau}_n$ have been constructed as in the induction hypothesis. Set

$$\eta_n := (\bar{u}_n(\bar{\tau}_n), \bar{v}_n(\bar{\tau}_n)),$$

which is $\bar{\mathcal{F}}_{n, \bar{\tau}_n}$ -measurable, since $(\bar{u}_n, \bar{v}_n)$ is $(\bar{\mathcal{F}}_{n, t})$ -adapted with continuous paths and $\bar{\tau}_n$ is a $(\bar{\mathcal{F}}_{n, t})$ -stopping time. Since $\bar{\tau}_n$ is by construction the endpoint of the n -th truncated piece (before any linear extension is entered), $\eta_n = (u_n(\tau_n^*), v_n(\tau_n^*))$, so nonnegativity of η_n follows from the nonnegativity of u_n, v_n established in Step II, and $\mu_n := \text{Law}(\eta_n)$ satisfies the integrability required in Assumption 4.5 by the moment bound of Proposition 4.48 for u_n together with Lemma 4.11 for v_n .

By Proposition 4.48 (truncation level $n+1$, initial law μ_n) there exist a probability space $\mathfrak{A}_{n+1} = (\Omega_{n+1}, \mathcal{F}_{n+1}, (\mathcal{F}_{n+1, t})_{t \in [0, T]}, \mathbb{P}_{n+1})$, a cylindrical Wiener process $\mathcal{W}_{n+1}$ on $\mathcal{H}$ over $\mathfrak{A}_{n+1}$, and a pair (u_{n+1}, v_{n+1}) that is a martingale solution to (4.6) (level $n+1$) with $(u_{n+1}(0), v_{n+1}(0))$ of law μ_n ; in general only $(u_{n+1}(0), v_{n+1}(0)) \stackrel{d}{=} \eta_n$, not pathwise. We therefore apply the Transfer Lemma (Lemma 4.61) with $\mathfrak{A}_1 := \bar{\mathfrak{A}}_n$, $\sigma := \bar{\tau}_n$, $(\bar{u}, \bar{v}) := (\bar{u}_n, \bar{v}_n)$, $\mathfrak{A}_2 := \mathfrak{A}_{n+1}$, obtaining a complete filtered probability space

$$\hat{\mathfrak{A}}_{n+1} = (\hat{\Omega}_{n+1}, \hat{\mathcal{F}}_{n+1}, (\hat{\mathcal{F}}_{n+1, t})_{t \in [0, T]}, \hat{\mathbb{P}}_{n+1}), \quad \hat{\Omega}_{n+1} = \bar{\Omega}_n \times X_{n+1},$$

with canonical projection $\pi_n: \hat{\Omega}_{n+1} \rightarrow \bar{\Omega}_n$, $(\pi_n)_* \hat{\mathbb{P}}_{n+1} = \bar{\mathbb{P}}_n$, a cylindrical Wiener process $\hat{\mathcal{W}}_{n+1}$ on $\mathcal{H}$, and a pair $(\hat{u}_{n+1}, \hat{v}_{n+1})$, martingale solution to (4.6) (level $n+1$), such that $\hat{\mathbb{P}}_{n+1}$ -a.s.

$$(\hat{u}_{n+1}(0), \hat{v}_{n+1}(0)) = \eta_n, \quad \hat{\mathcal{W}}_{n+1} \text{ independent of } \pi_n^{-1}(\bar{\mathcal{F}}_n) \supseteq \pi_n^{-1}(\bar{\mathcal{F}}_{n, \bar{\tau}_n}).$$

(We tacitly identify every $\bar{\Omega}_n$ -measurable object with its pullback along π_n from here on.)

Define, analogously to Step I,

$$\begin{aligned}\tau_{n+1}^1 &:= \inf \{s \geq 0 \mid \|\hat{u}_{n+1}(s)\|_{L^1_{\mathcal{O}}} \geq n+1\}, & \tau_{n+1}^2 &:= \inf \{s \geq 0 \mid \|\nabla \hat{v}_{n+1}(s)\|_{L^2_{\mathcal{O}}}^2 \geq n+1\}, \\ \tau_{n+1}^3 &:= \inf \{s \geq 0 \mid \int_0^s \|\nabla \hat{v}_{n+1}(r)\|_{H^1_{\mathcal{O}}}^2 dr \geq n+1\}, & \tau_{n+1}^* &:= \min(\tau_{n+1}^1, \tau_{n+1}^2, \tau_{n+1}^3, T),\end{aligned}$$

and set $\bar{\tau}_{n+1} := \bar{\tau}_n + \tau_{n+1}^*$. As in the base case, for $s \leq \tau_{n+1}^*$ the cutoff in (4.6) equals 1, so $(\hat{u}_{n+1}, \hat{v}_{n+1})$ solves the original (untruncated) system on $[0, \tau_{n+1}^*]$. Let (y_{n+1}, z_{n+1}) be the solution to System (4.24) for $\sigma = \bar{\tau}_{n+1}$, $\bar{\mathcal{W}} = \bar{\mathcal{W}}_{n+1}$ (defined below), and $(y_0, z_0) = (\hat{u}_{n+1}(\tau_{n+1}^*), \hat{v}_{n+1}(\tau_{n+1}^*))$.

Set $\bar{\mathfrak{A}}_{n+1} := \hat{\mathfrak{A}}_{n+1}$,

$$\bar{\mathcal{F}}_{n+1,t} := \overline{\sigma\left(\bar{\mathcal{F}}_{n,t \wedge \bar{\tau}_n} \cup \sigma((\hat{u}_{n+1}, \hat{v}_{n+1}, \hat{\mathcal{W}}_{n+1})(s) : s \leq (t - \bar{\tau}_n)^+)\right)}^{\bar{\mathbb{P}}_{n+1}}, \quad t \in [0, T],$$

$$\bar{\mathcal{W}}_{n+1}(t) := \begin{cases} \bar{\mathcal{W}}_n(t), & t \in [0, \bar{\tau}_n], \\ \bar{\mathcal{W}}_n(\bar{\tau}_n) + \hat{\mathcal{W}}_{n+1}(t - \bar{\tau}_n), & t \in (\bar{\tau}_n, T], \end{cases}$$

and

$$(\bar{u}_{n+1}(t), \bar{v}_{n+1}(t)) := \begin{cases} (\bar{u}_n(t), \bar{v}_n(t)), & t \in [0, \bar{\tau}_n], \\ (\hat{u}_{n+1}(t - \bar{\tau}_n), \hat{v}_{n+1}(t - \bar{\tau}_n)), & t \in [\bar{\tau}_n, \bar{\tau}_{n+1}], \\ (y_{n+1}(t - \bar{\tau}_{n+1}), z_{n+1}(t - \bar{\tau}_{n+1})), & t \in [\bar{\tau}_{n+1}, T]. \end{cases}$$

Continuity of $(\bar{u}_{n+1}, \bar{v}_{n+1})$ at $t = \bar{\tau}_n$ holds because $(\hat{u}_{n+1}(0), \hat{v}_{n+1}(0)) = \eta_n = (\bar{u}_n(\bar{\tau}_n), \bar{v}_n(\bar{\tau}_n))$ pathwise, by Lemma 4.61(i); across $t = \bar{\tau}_{n+1}$ it holds because (y_0, z_0) was chosen to equal $(\hat{u}_{n+1}(\tau_{n+1}^*), \hat{v}_{n+1}(\tau_{n+1}^*))$ exactly. That $\bar{\mathcal{W}}_{n+1}$ is a genuine cylindrical Wiener process over $\bar{\mathfrak{A}}_{n+1}$ relative to $(\bar{\mathcal{F}}_{n+1,t})$ follows by the same gluing argument as for the shifted Wiener process in the construction of (y_1, z_1) : $\bar{\mathcal{W}}_n$ has independent increments relative to $(\bar{\mathcal{F}}_{n,t \wedge \bar{\tau}_n})$ on $[0, \bar{\tau}_n]$ by the induction hypothesis, and $\hat{\mathcal{W}}_{n+1}$, started afresh at $\bar{\tau}_n$, has independent increments and is independent of $\bar{\mathcal{F}}_{n,\bar{\tau}_n}$ – here by Lemma 4.61(ii) rather than by the strong Markov property, but the concatenation argument itself is otherwise identical. Finally, $(\bar{u}_{n+1}, \bar{v}_{n+1})$ is a martingale solution to (4.6) on $[0, \bar{\tau}_{n+1}]$ over $\bar{\mathfrak{A}}_{n+1}$: on $[0, \bar{\tau}_n]$ this is the induction hypothesis, and on $[\bar{\tau}_n, \bar{\tau}_{n+1}]$ it is the martingale-solution property of $(\hat{u}_{n+1}, \hat{v}_{n+1})$ from Lemma 4.61, transported through the time shift $t \mapsto t - \bar{\tau}_n$. This completes the induction step. $\square$

From $(\bar{\mathfrak{A}}_n)_{n \in \mathbb{N}}$ to a single fixed probability space. The induction just carried out produces, for every n , a genuinely different probability space $\bar{\mathfrak{A}}_n = (\bar{\Omega}_n, \bar{\mathcal{F}}_n, \bar{\mathbb{P}}_n)$: by construction $\bar{\Omega}_{n+1} = \bar{\Omega}_n \times X_{n+1}$, so $\bar{\Omega}_n = \Omega_1 \times X_2 \times \cdots \times X_n$ is a growing finite product, related by the canonical projections $\pi_n: \bar{\Omega}_{n+1} \rightarrow \bar{\Omega}_n$ with $(\pi_n)_* \bar{\mathbb{P}}_{n+1} = \bar{\mathbb{P}}_n$ (this is the “tacit identification” convention introduced above). This is exactly a projective (inverse) system of probability spaces, and since each X_k is Polish and each $\bar{\mathbb{P}}_k$ a Borel probability measure, Kolmogorov’s extension theorem yields a projective limit: a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega := \Omega_1 \times \prod_{k=2}^{\infty} X_k$, together with projections $\pi^{(n)}: \Omega \rightarrow \bar{\Omega}_n$ onto the first n factors satisfying $(\pi^{(n)})_* \mathbb{P} = \bar{\mathbb{P}}_n$ for every n . Extending the tacit-identification convention from the single step π_n used above to $\pi^{(n)}$, we henceforth regard every object defined on some $\bar{\Omega}_n$ as defined on the fixed Ω via pullback along $\pi^{(n)}$, with $\mathbb{P}(B) = \bar{\mathbb{P}}_n(B)$ for every $\bar{\mathcal{F}}_n$ -measurable B . *This is the point, and the only point, in the construction at which the probability space becomes fixed once and for all*, with n from here on affecting only the solution pair $(\bar{u}_n, \bar{v}_n)$ and the stopping time $\bar{\tau}_n$, not the underlying $(\Omega, \mathcal{F}, \mathbb{P})$; it is what justifies writing bare $\Omega, \mathbb{P}$ (rather than $\Omega_n, \mathbb{P}_n$) from Step V onward, and the identity $\mathbb{P}(A_n) = \bar{\mathbb{P}}_n(A_n)$ used there.

4.2.5 Step IV: A Lyapunov Estimate Adapted to Consumption

The Lyapunov functional used by [HMT22a] for the production model,

$$W(u, v) = \int_{\mathcal{O}} u \log u - \rho \int_{\mathcal{O}} uv + C_1 \|\nabla v\|_{L^2}^2 + C_2 \|v\|_{L^2}^2,$$

carries a cross term $-\rho \int_{\mathcal{O}} uv$ tailored to compensate the destabilizing production nonlinearity $+\beta u$ in the second equation. In the consumption model considered here this term is absent. Following the deterministic energy functional of Section 3.3, we instead replace it by the Fisher information $2 \int_{\mathcal{O}} |\nabla \sqrt{v}|^2$.

In this section we want to bound the following functional in expectation

$$\mathcal{F}(u, v)(t) := \int_{\mathcal{O}} u \log u + 2 \int_{\mathcal{O}} |\nabla \sqrt{v}|^2. \quad (4.25)$$

Note that in this consumption system we replaced $-\rho \int_{\mathcal{O}} uv$ by

$$2 \int_{\mathcal{O}} |\nabla \sqrt{v}|^2 = \frac{1}{2} \int_{\mathcal{O}} \frac{|\nabla v|^2}{v}.$$

Additionally we introduce

$$\mathcal{D}(u, v)(t) := \int_{\mathcal{O}} \frac{|\nabla u|^2}{u} + \int_{\mathcal{O}} v |D^2 \log v|^2 + \int_{\mathcal{O}} u \frac{|\nabla v|^2}{2v}, \quad (4.26)$$

with the goal to show

$$\mathbb{E}[\mathcal{F}(\bar{u}_n, \bar{v}_n)(t)] \leq \mathbb{E}\left[\mathcal{F}(\bar{u}_n, \bar{v}_n)(t) + \int_0^t \mathcal{D}(\bar{u}_n, \bar{v}_n)(s) ds\right] \leq C$$

for some constant $C > 0$. The first inequality is obvious due to $\mathcal{D}(u, v) \geq 0$ when u and v are non-negative.

Since $(\bar{u}_n, \bar{v}_n)$ is neither regular enough to use the Itô Formula nor regular enough for ∇u to even be defined properly, one needs to use an approximation $(u^\varepsilon, v^\varepsilon)$ which is obtained by one of the following ways.

- (i) Using $(u^\varepsilon, v^\varepsilon) = (Y_\varepsilon \bar{u}_n, Y_\varepsilon \bar{v}_n)$ with $Y_\varepsilon = (I - \varepsilon \Delta)^{-1}$ being the Yosida Approximation which solves

$$\begin{aligned} u^\varepsilon(t) &= u_0^\varepsilon + \int_0^t (\Delta u^\varepsilon(s) + \gamma u^\varepsilon - \Psi_n(s) Y_\varepsilon \operatorname{div}(\bar{u}_n \nabla \bar{v}_n)(s)) ds + \int_0^t u^\varepsilon(s) dW(s), \\ v^\varepsilon(t) &= v_0^\varepsilon + \int_0^t (\Delta v^\varepsilon - Y_\varepsilon(\bar{u}_n \nabla \bar{v}_n)) ds, \end{aligned}$$

$\mathbb{P}_n$ -a.s. for all $t \in [0, T]$.

This ansatz does not work because the term $\langle \phi'_\delta(u^\varepsilon), -Y_\varepsilon \operatorname{div}(\bar{u}_n \nabla \bar{v}_n) \rangle_{\mathcal{O}}$, with ϕ_δ being an extension of $r \log r$ to the whole space $\mathbb{R}$, can not be controlled by $\phi_\delta(u^\varepsilon)$ due to the fact that div and ∇ do not commute with Y_ε in the scalar product.

(ii) Using the approximative system

$$\begin{cases} du^\varepsilon = (\Delta_\varepsilon u^\varepsilon + \gamma u^\varepsilon - \Psi_n \operatorname{div}_\varepsilon(u^\varepsilon \nabla v^\varepsilon)) dt + u^\varepsilon dW(t), \\ \partial_t v^\varepsilon = \Delta_\varepsilon v^\varepsilon - u^\varepsilon v^\varepsilon \end{cases}$$

where $\Delta_\varepsilon = \frac{1}{\varepsilon}((I - \varepsilon \Delta)^{-1} - I)$ and $\operatorname{div}_\varepsilon(w) := \operatorname{div}(\eta_\varepsilon \star w)$ where $\eta_\varepsilon(x) := \varepsilon^{-1} \eta(\frac{x}{\varepsilon}) \xrightarrow{\varepsilon \rightarrow 0} \delta(x)$ with η being the standard mollifier, i.e., $\eta \in C_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \eta = 1$.

Unlike in (i), where it is obvious to show $(u^\varepsilon, v^\varepsilon) \rightarrow (\bar{u}_n, \bar{v}_n)$ for $\varepsilon \rightarrow 0$ in some sense, here we have to prove that. This is (if possible) not easy.²

(iii) Using the approximative system

$$\begin{cases} du^\varepsilon = (\Delta u^\varepsilon + \gamma u^\varepsilon - \Psi_n \operatorname{div}(u^\varepsilon F'_\varepsilon(u^\varepsilon) \nabla v^\varepsilon)) dt + u^\varepsilon dW(t), \\ \partial_t v^\varepsilon = \Delta v^\varepsilon - F_\varepsilon(u^\varepsilon) v^\varepsilon, \end{cases}$$

for $F_\varepsilon(r) := \frac{1}{\varepsilon} \log(1 + \varepsilon r)$ and $\varepsilon \in (0, 1)$ as in [TW12], cf. also the regularized deterministic system (3.7).

The authors of this paper have shown everything needed (existence of the approximative system, convergence, energy estimates, etc.) for the deterministic version, but even to show existence for the version with multiplicative noise is not easy (if possible).

There may be more possible ways to tackle this problem but (iii) might be the most promising. Since there is simply not enough time and space in this thesis to further discuss this, we will just skip the approximation and work directly with $(\bar{u}_n, \bar{v}_n)$ even if this is not rigorous (especially because we do not have $\bar{u}_n, \bar{v}_n > 0$).

We now differentiate between the case that $t \in [0, \bar{\tau}_n]$ where $(\bar{u}_n, \bar{v}_n)$ follows the original system ($\Psi_n = 1$) and the case that $t \in (\bar{\tau}_n, T]$, where $(\bar{u}_n, \bar{v}_n)$ follows the simple heat equations. Hence the upper integral bound $t \wedge \bar{\tau}_n$.

We will be writing (u, v) instead of $(\bar{u}_n, \bar{v}_n)$ and omit (t, x) and (t, s) for simplicity. Defining $f(r) = r \log r$ we see $f'(r) = \log r + 1$ and $f''(r) = \frac{1}{r}$. First we work with the term $u \log u$.

Lemma 4.63. *There exists $C = C(T, |\mathcal{O}|) > 0$, independent of n , such that*

$$\begin{aligned} \int_{\mathcal{O}} (u \log u)(t \wedge \bar{\tau}_n) dx + \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} \frac{|\nabla u|^2}{u} dx ds &\leq \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} [\nabla u \nabla v + C u \log u] dx ds \\ &\quad + C \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} (u \log u + 1) dx dW(s) + C. \end{aligned}$$

²The authors of [HMT22a] state that they have chosen this way but cite a paper using (i) in a system where the divergence term does not appear. Also they state that $\|\Delta_\varepsilon - \Delta\|_{L(H^{\nu+\delta}, H^\delta)} \rightarrow 0$ as $\varepsilon \rightarrow 0$ for all $\delta \in \mathbb{R}$ and $\nu > 0$ but the statement is only true for $\nu > 2$.

Proof. Using the Itô Formula we obtain that

$$\begin{aligned} \int_{\mathcal{O}} f(u(t \wedge \bar{\tau}_n)) \, dx - \int_{\mathcal{O}} f(u_0) \, dx &= \int_0^{t \wedge \bar{\tau}_n} \langle f'(u), \Delta u + \gamma u - \operatorname{div}(u \nabla v) \rangle_{\mathcal{O}} \, ds \\ &\quad + \int_0^{t \wedge \bar{\tau}_n} \langle f'(u), u \rangle_{\mathcal{O}} \, d\mathcal{W}(s) \\ &\quad + \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_0^{t \wedge \bar{\tau}_n} \langle f''(u), |u e_k^{(\delta_1)}|^2 \rangle_{\mathcal{O}} \, ds, \end{aligned}$$

which by partial integration, $\sum_{k \in \mathbb{Z}} |e_k^{(\delta_1)}|^2 = \gamma_1$ and plugging in f leads to, writing $\tilde{t} := t \wedge \bar{\tau}_n$,

$$\begin{aligned} \int_{\mathcal{O}} (u \log u)(\tilde{t}) \, dx - \int_{\mathcal{O}} u_0 \log u_0 &= \int_0^{\tilde{t}} \int_{\mathcal{O}} \left[-\frac{|\nabla u|^2}{u} + \gamma u (\log u + 1) + \frac{\gamma_1}{2} u + \frac{u \nabla u \nabla v}{u} \right] \, dx \, ds \\ &\quad + \int_0^{\tilde{t}} \int_{\mathcal{O}} u (\log u + 1) \, dx \, d\mathcal{W}(s). \end{aligned}$$

This together with $r \leq r \log r + 1$ for $r \geq 0$ and $u_0 \in L^2(\mathcal{O}) \hookrightarrow L \log L(\mathcal{O})$ yields the claim. $\square$

Next in line is the term with $|\nabla \sqrt{v}|^2$. For this we need to define another function

Definition 4.64. We define

$$g(v, w) := \frac{w^2}{2v},$$

with its derivative

$$Dg(v, w) = \left(-\frac{w^2}{2v^2} \quad \frac{w}{v} \right).$$

Lemma 4.65. *The following identity is given for g*

$$Dg(v, \nabla v)[h] = \int_{\mathcal{O}} \left(\frac{|\nabla v|^2}{2v^2} - \frac{\Delta v}{v} \right) h.$$

Proof. Linearizing $v \rightarrow v + \varepsilon h$ and $w = \nabla v \rightarrow \nabla v + \varepsilon \nabla h$ yields

$$Dg(v, \nabla v)[h] = \int_{\mathcal{O}} Dg(v, \nabla v) \begin{pmatrix} h \\ \nabla h \end{pmatrix} = - \int_{\mathcal{O}} \frac{|\nabla v|^2}{2v^2} h + \int_{\mathcal{O}} \frac{\nabla v}{v} \cdot \nabla h.$$

Integrating by parts in the second integral yields the claim because

$$-\operatorname{div} \frac{\nabla v}{v} = -\frac{\Delta v}{v} + \frac{|\nabla v|^2}{v^2}. \quad \square$$

Lemma 4.66. *There exists $C(|\mathcal{O}|) > 0$, independent of n , such that*

$$\int_{\mathcal{O}} \frac{|\nabla v|^2}{2v} (t \wedge \bar{\tau}_n) \, dx + \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} \left[v |D^2 \log v|^2 + \frac{u |\nabla v|^2}{2v} \right] \, dx \, ds \leq - \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} \nabla u \nabla v \, dx \, ds + C.$$

Proof. We can easily see that

$$\begin{aligned} \int_{\mathcal{O}} [g(v, \nabla v)(t \wedge \bar{\tau}_n) - g(v_0, \nabla v_0)] \, dx &= \int_0^{t \wedge \bar{\tau}_n} Dg(v, \nabla v)[\partial_t v] \, ds = \int_0^{t \wedge \bar{\tau}_n} \left\langle \frac{|\nabla v|^2}{2v^2} - \frac{\Delta v}{v}, \Delta v - uv \right\rangle \\ &= \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} \left[\frac{|\nabla v|^2 \Delta v}{2v^2} + u \Delta v - \frac{u |\nabla v|^2}{2v} - \frac{|\Delta v|^2}{v} \right] \, dx \, ds, \end{aligned}$$

where the last addend can be written, by identity (2.18) of Lemma 2.21 applied with $w = v$, as

$$- \int_{\mathcal{O}} \frac{|\Delta v|^2}{v} \, dx = \int_{\mathcal{O}} \left[-\frac{|D^2 v|^2}{v} - \frac{3}{2} \frac{|\nabla v|^2 \Delta v}{v^2} + \frac{|\nabla v|^4}{v^3} \right] \, dx + \frac{1}{2} \int_{\partial \mathcal{O}} \frac{1}{v} \partial_\nu |\nabla v|^2 \, dx.$$

Substituting this into the bracket above and regrouping the remaining terms

$$\frac{|\nabla v|^2 \Delta v}{2v^2} - \frac{3}{2} \frac{|\nabla v|^2 \Delta v}{v^2} = -\frac{|\nabla v|^2 \Delta v}{v^2}$$

shows that the bracket equals

$$u \Delta v - \frac{u |\nabla v|^2}{2v} - \left[\frac{|D^2 v|^2}{v} - \frac{|\nabla v|^4}{v^3} + \frac{|\nabla v|^2 \Delta v}{v^2} \right],$$

and the term in square brackets is exactly $v |D^2 \log v|^2$ by identity (2.17) of Lemma 2.21. Since $\langle u, \Delta v \rangle_{\mathcal{O}} = -\langle \nabla u, \nabla v \rangle_{\mathcal{O}}$, we obtain

$$\begin{aligned} \int_{\mathcal{O}} [g(v, \nabla v)(t \wedge \bar{\tau}_n) - g(v_0, \nabla v_0)] \, dx &= \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} \left[-\nabla u \nabla v - \frac{u |\nabla v|^2}{2v} - v |D^2 \log v|^2 \right] \, dx \, ds \\ &\quad + \frac{1}{2} \int_0^{t \wedge \bar{\tau}_n} \int_{\partial \mathcal{O}} \frac{1}{v} \partial_\nu |\nabla v|^2 \, dx \, ds. \end{aligned}$$

Since $\mathcal{O} \subset \mathbb{R}$ is an interval and hence convex, Lemma 2.20 applies and gives $\partial_\nu |\nabla v|^2|_{\partial \mathcal{O}} \leq 0$. Finally $\sqrt{v_0} \in H^1(\mathcal{O})$ (Assumption 4.5(d)) yields the claim. $\square$

Corollary 4.67. *There exists $C = C(T, |\mathcal{O}|) > 0$, independent of n , such that*

$$\begin{aligned} \mathcal{F}(u, v)(t \wedge \bar{\tau}_n) + \int_0^{t \wedge \bar{\tau}_n} \mathcal{D}(u, v)(s) \, ds &\leq C \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} u \log u \, dx \, ds \\ &\quad + C \int_0^{t \wedge \bar{\tau}_n} \int_{\mathcal{O}} u \log u \, dx \, d\mathcal{W}(s) \\ &\quad + C + C\mathcal{W}(t \wedge \bar{\tau}_n). \end{aligned}$$

Proof. Lemma 4.63, Lemma 4.66 and $\int_0^t d\mathcal{W}(s) = \mathcal{W}(t)$ because $\mathcal{W}(0) = 0$. $\square$

We still have to deal with the stochastic integral.

Lemma 4.68. *There exists $C > 0$ such that*

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} \left| \int_0^s \int_{\mathcal{O}} f(u) \, dx \, d\mathcal{W}(\sigma) \right| \right] \leq C \mathbb{E}_n \left[\left(\int_0^t \left(\int_{\mathcal{O}} f(u) \, dx \right)^2 \, ds \right)^{\frac{1}{2}} \right].$$

Proof. Let us call $\xi_k(s) := \langle f(u(s)), e_k^{(\delta_1)} \rangle_{\mathcal{O}}$ and $X(s) := \int_{\mathcal{O}} f(u(s)) \, dx$. By definition of $\mathcal{W}$ we obtain

$$M(t) := \int_{\mathcal{O}} \int_0^t f(u(s, x)) \, d\mathcal{W}(s, x) \, dx = \sum_{k \in \mathbb{Z}} \int_0^t \xi_k(s) \, d\beta_k^{(1)}(s)$$

which using the BDG Theorem 2.36 leads directly to

$$\mathbb{E}_n \left[\sup_{s \in [0, t]} |M(s)| \right] \leq C_B \mathbb{E}_n \left[\left(\sum_{k \in \mathbb{Z}} \int_0^t |\xi_k(s)|^2 \, ds \right)^{\frac{1}{2}} \right].$$

We also can estimate $|\xi_k(s)|$ by

$$|\xi_k(s)| = \left| \int_{\mathcal{O}} f(u) \overbrace{e_k^{(\delta_1)}}^{=\eta_k e_k} \, dx \right| \leq \sqrt{2} \eta_k |X(s)|,$$

which by inserting into the inequality above and using $\sum_{k \in \mathbb{Z}} \eta_k^2 = \gamma_1$ (Assumption 4.3) yields the claim. $\square$

Lemma 4.69. *There exists $C = C(T, |\mathcal{O}|) > 0$, independent of n , such that for all $t \in [0, T]$*

$$\mathbb{E}_n \left[\sup_{s \in [0, t \wedge \bar{\tau}_n]} \int_{\mathcal{O}} \left[\bar{u}_n(s) \log \bar{u}_n(s) + |\nabla \sqrt{\bar{v}_n}(s)|^2 \right] \, dx \right] \leq C.$$

Proof. Obviously $u(t) \log u(t) \leq \mathcal{F}(u, v)(t)$ and $\mathbb{E}_n[\mathcal{W}(t)] = 0$ for all $t \in [0, T]$. Taking the supremum and expectation in the inequality of Corollary 4.67 and using Lemma 4.68 we obtain

$$\begin{aligned} \mathbb{E}_n \left[\sup_{s \in [0, t \wedge \bar{\tau}_n]} \mathcal{F}(u, v)(s) \right] &\leq C_1 \mathbb{E}_n \left[\int_0^{t \wedge \bar{\tau}_n} \sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma) \, d\sigma \right] \\ &\quad + C_2 \mathbb{E}_n \left[\left(\int_0^{t \wedge \bar{\tau}_n} \sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma)^2 \, ds \right)^{\frac{1}{2}} \right] + C_3 \end{aligned}$$

Using that for $s \leq t$

$$\sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma)^2 \leq \sup_{\sigma \in [0, t]} \mathcal{F}(u, v)(\sigma)^2,$$

we can find that with Young

$$\begin{aligned} \left(\int_0^t \sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma)^2 \, ds \right)^{\frac{1}{2}} &\leq \left(\sup_{\sigma \in [0, t]} \mathcal{F}(u, v)(\sigma) \right)^{\frac{1}{2}} \left(\int_0^t \sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma) \, ds \right)^{\frac{1}{2}} \\ &\leq \alpha \sup_{\sigma \in [0, t]} \mathcal{F}(u, v)(\sigma) + C_Y(\alpha) \int_0^t \sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma) \, ds. \end{aligned}$$

Now we can choose α small enough so the first addend will be absorbed in the left side. Hence with Fubini we arrive at

$$\mathbb{E}_n \left[\sup_{s \in [0, t \wedge \bar{\tau}_n]} \mathcal{F}(u, v)(s) \right] \leq C_4 \int_0^{t \wedge \bar{\tau}_n} \mathbb{E}_n \left[\sup_{\sigma \in [0, s]} \mathcal{F}(u, v)(\sigma) \right] \, d\sigma + C_3$$

which with Gronwall yields the claim. $\square$

For $t \in (\bar{\tau}_n, T]$ the estimate is easier: Since $-\operatorname{div}(u\nabla v)$ disappears from the first equation and $-uv$ from the second one, the $\nabla u \nabla v$ -term vanishes from both estimates as well as the term $\frac{u|\nabla v|^2}{v}$ from the left hand side of the estimate in Lemma 4.66. Hence we finish this section with the following result.

Proposition 4.70. *There exists a constant $C = C(T, |\mathcal{O}|) > 0$, such that for every $n \in \mathbb{N}$*

$$\mathbb{E}_n \left[\sup_{t \in [0, T]} \|\bar{u}_n(t)\|_{L \log L_{\mathcal{O}}} \right] + \mathbb{E}_n \left[\sup_{t \in [0, T]} \|\nabla \bar{v}_n\|_{L^2_{\mathcal{O}}}^2 \right] + \mathbb{E}_n \left[\int_0^T \|\nabla \bar{v}_n(t)\|_{H^1_{\mathcal{O}}}^2 dt \right] \leq C.$$

Proof. It is easy to verify that $r \log(r+1) \leq r \log r + \max(1, \log 2 + \frac{1}{e})$, hence by Lemma 4.69 (and the argument above for $t \in (\bar{\tau}_n, T]$) there exists $C_1 > 0$ such that

$$\mathbb{E}_n \left[\sup_{t \in [0, T]} \|\bar{u}_n(t)\|_{L \log L_{\mathcal{O}}} \right] \leq \mathbb{E}_n \left[\sup_{t \in [0, T]} \int_{\mathcal{O}} \bar{u}_n \log \bar{u}_n dx \right] \leq C_1.$$

Since $r \log r \geq -\frac{1}{e}$, $\mathcal{F} \geq -\frac{|\mathcal{O}|}{e}$ holds pointwise. Because $\bar{u}_n, \bar{v}_n \geq 0$ (Propositions 4.55 and 4.56, transported through the gluing of Section 4.2.4) we see $\partial_t \bar{v}_n \leq \Delta \bar{v}_n$ and hence by the Hopf Maximum Principle for Neumann boundary condition $\|\bar{v}_n(t)\|_{L^\infty_{\mathcal{O}}} \leq \|v_0\|_{L^\infty_{\mathcal{O}}}$ for all t . Using $|\nabla \bar{v}_n|^2 = 4v|\nabla \sqrt{\bar{v}_n}|^2$ we obtain

$$\|\nabla \bar{v}_n(t)\|_{L^2_{\mathcal{O}}}^2 \leq 4\|v_0\|_{L^\infty_{\mathcal{O}}} \int_{\mathcal{O}} |\nabla \sqrt{\bar{v}_n(t)}|^2 dx \leq 4\|v_0\|_{L^\infty_{\mathcal{O}}} \cdot \frac{1}{2} \left(\mathcal{F}(u, v)(t) + \frac{|\mathcal{O}|}{e} \right) \quad \forall t \in [0, T].$$

Since $\|v_0\|_{L^\infty} \leq K$ $\mathbb{P}$ -a.s. by Assumption 4.5(e), taking the supremum and expectation yields the estimate for the second term in the claim.

By [Win12, Lemma 3.3] (with $h(s) = s$) we have $\int_{\mathcal{O}} \frac{|\nabla w|^4}{w^3} \leq C_5 \int_{\mathcal{O}} w |D^2 \log w|^2$. With $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ for all $a, b \in \mathbb{R}$ ($\iff \frac{1}{2}(a-2b)^2 \geq 0$) applied to the norms of $D^2 \log w = \frac{D^2 w}{w} - \frac{\nabla w \otimes \nabla w}{w^2}$ we obtain

$$|D^2 \log w|^2 = \left| \frac{D^2 w}{w} - \frac{\nabla w \otimes \nabla w}{w^2} \right|^2 \geq \frac{|D^2 w|^2}{2w^2} - \frac{|\nabla w|^4}{w^4} \implies \int_{\mathcal{O}} \frac{|D^2 w|^2}{w} \leq C_6 \int_{\mathcal{O}} w |D^2 \log w|^2.$$

Hence we arrive at

$$\int_{\mathcal{O}} |D^2 \bar{v}_n|^2 \leq \|v_0\|_{L^\infty_{\mathcal{O}}} \int_{\mathcal{O}} \frac{|D^2 \bar{v}_n|^2}{\bar{v}_n} \leq C_6 \int_{\mathcal{O}} \bar{v}_n |D^2 \log \bar{v}_n|^2.$$

Taking the expectation of the inequality in Corollary 4.67 and using Lemma 4.68 and the proof of Lemma 4.69 shows

$$\mathbb{E}_n \left[\int_0^T \int_{\mathcal{O}} \bar{v}_n |D^2 \log \bar{v}_n|^2 dx ds \right] \leq C_4 \mathbb{E}_n \left[\int_0^T \mathcal{F}(\bar{u}_n, \bar{v}_n)(t) dt \right] + C_4 + \frac{|\mathcal{O}|}{e} \leq C_5.$$

With $\|\nabla \bar{v}_n\|_{H^1_{\mathcal{O}}}^2 \leq \|\nabla \bar{v}_n\|_{L^2_{\mathcal{O}}}^2 + \|D^2 \bar{v}_n\|_{L^2_{\mathcal{O}}}^2$ that yields the claim. $\square$

A remark on rigor is in order. The computation in the proof of Lemma 4.66 differen-

tiates $g(v, \nabla v)$ in time and integrates the identity of Lemma 2.21 by parts, treating v (in the application, $\bar{v}_n$) as if it were classical. The regularity of $\bar{v}_n$ established in Step III (Section 4.2.4) does not, by itself, justify this computation directly. A fully rigorous argument would probably mollify (u, v) in the way described in (iii) above.

4.2.6 Step V: Globalization

Finally we want to show that there exists a solution on the whole interval $[0, T]$. For this we define

$$A_n := \{\omega \in \Omega : \bar{\tau}_n(\omega) \geq T\}.$$

We want to show $\mathbb{P}(A_n) \rightarrow 1$ as $n \rightarrow \infty$.

First we need the following lemma, which helps when gluing the solutions together. We will omit the proof here.

Lemma 4.71. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, E a Polish space, $(\sigma_n)_{n \in \mathbb{N}}$ a non-decreasing sequence of $(\mathcal{F}_t)$ -stopping times with values in $[0, T]$, and $(X_n)_{n \in \mathbb{N}}$, $X_n : [0, T] \times \Omega \rightarrow E$, a sequence of $(\mathcal{F}_t)$ -adapted processes with $\mathbb{P}$ -a.s. continuous paths satisfying*

$$X_{n+1}(t) = X_n(t) \quad \text{for all } t \in [0, \sigma_n], \quad \mathbb{P}\text{-a.s.}, \quad n \in \mathbb{N}. \quad (4.27)$$

Set $\tilde{\Omega} := \bigcup_{n \in \mathbb{N}} \{\sigma_n \geq T\}$ and, for $\omega \in \tilde{\Omega}$, $n(\omega) := \min \{n \in \mathbb{N} : \sigma_n(\omega) \geq T\}$. Define

$$X(t, \omega) := X_{n(\omega)}(t, \omega), \quad t \in [0, T], \quad \omega \in \tilde{\Omega}.$$

Then X has $\mathbb{P}$ -a.s. continuous paths and is $(\mathcal{F}_t)$ -progressively measurable on $[0, T] \times \tilde{\Omega}$.

Lemma 4.72. *On A_n , $(\bar{u}_n, \bar{v}_n)|_{[0, T]}$ solves (4.5).*

Proof. By the inductive construction in 4.2.4, $(\bar{u}_n, \bar{v}_n)$ solves (4.5) on $[0, \bar{\tau}_n]$. On A_n , $[0, T]$ is contained in this interval, so $(\bar{u}_n, \bar{v}_n)|_{[0, T]}$ is itself a solution of (4.5) on all of $[0, T]$ and we don't need the linear extension via the heat equations. Also $A_n \subseteq A_{n+1}$, since $\bar{\tau}_{n+1} = \bar{\tau}_n + \tau_{n+1}^* \geq \bar{\tau}_n$. $\square$

Lemma 4.73. *There exists a constant $C = C(T, |\mathcal{O}|)$, independent of n , such that*

$$\mathbb{P}(\mathcal{O} \setminus A_n) \leq \frac{C}{n}.$$

Proof. Since $\bar{\tau}_n \geq \tau_n^*$, we have $\{\bar{\tau}_n < T\} \subseteq \{\tau_n^* < T\}$. If $\tau_n^* < T$, then by right-continuity of (u_n, v_n) and the definition of $\tau_n^* = \min(\tau_n^1, \tau_n^2, \tau_n^3, T)$ as stopping time, at least one of

$$\|u_n(\tau_n^*)\|_{L^1_{\mathcal{O}}} = n, \quad \|\nabla v_n(\tau_n^*)\|_{L^2_{\mathcal{O}}}^2 = n, \quad \int_0^{\tau_n^*} \|\nabla v_n(s)\|_{H^1_{\mathcal{O}}}^2 ds = n,$$

holds. After the gluing and transfer to $(\bar{u}_n, \bar{v}_n)$ and an absolute time $\bar{\tau}_n \in [0, T]$ via

$$(u_n(s), v_n(s)) = (\bar{u}_n(\bar{\tau}_{n-1} + s), \bar{v}_n(\bar{\tau}_{n-1} + s)), \quad s \in [0, \tau_n^*]$$

we arrive at

$$\begin{aligned} \Omega \setminus A_n &\subseteq \{\tau_n^* < T\} \\ &\subseteq \left\{ \sup_{t \in [0, T]} \|\bar{u}_n(t)\|_{L^1_{\mathcal{O}}} \geq n \right\} \cup \left\{ \sup_{t \in [0, T]} \|\nabla \bar{v}_n(t)\|_{L^2_{\mathcal{O}}}^2 \geq n \right\} \cup \left\{ \int_0^T \|\nabla \bar{v}_n(t)\|_{H^1_{\mathcal{O}}}^2 dt \geq n \right\}. \end{aligned}$$

Using Markov's inequality on each of the three events and the embedding $L \log L(\mathcal{O}) \hookrightarrow L^1(\mathcal{O})$, by Proposition 4.70 we get, writing $\mathbb{E}_n := \mathbb{E}_{\bar{\mathbb{P}}_n}$ for the expectation on $\bar{\Omega}_n$ used there, which equals $\mathbb{E}$ on $\bar{\mathcal{F}}_n$ -measurable objects by the identification $\mathbb{P}(B) = \bar{\mathbb{P}}_n(B)$ established above,

$$\begin{aligned} \mathbb{P}(\Omega \setminus A_n) &= \bar{\mathbb{P}}_n(\bar{\Omega}_n \setminus A_n) \\ &\leq \frac{1}{n} \left(\mathbb{E}_n \left[\sup_{t \in [0, T]} \|\bar{u}_n(t)\|_{L \log L_{\mathcal{O}}} \right] + \mathbb{E}_n \left[\sup_{t \in [0, T]} \|\nabla \bar{v}_n(t)\|_{L^2_{\mathcal{O}}}^2 \right] + \mathbb{E}_n \left[\int_0^T \|\nabla \bar{v}_n(t)\|_{H^1_{\mathcal{O}}}^2 dt \right] \right) \\ &\leq \frac{C}{n}. \end{aligned} \quad \square$$

Proposition 4.74. *There exists a measurable set $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that on $\tilde{\Omega}$ a martingale solution (u, v) of (4.5) exists.*

Proof. By Lemma 4.73, $\mathbb{P}(\Omega \setminus A_n) \rightarrow 0$ as $n \rightarrow \infty$. With $A_n \subseteq A_{n+1}$ from Lemma 4.72, $\tilde{\Omega} := \bigcup_{n \in \mathbb{N}} A_n$ satisfies $\mathbb{P}(\tilde{\Omega}) = \lim_{n \rightarrow \infty} \mathbb{P}(A_n) = 1$. We set $A_0 := \emptyset$ and define

$$(u, v) := (\bar{u}_n, \bar{v}_n)|_{[0, T]} \quad \text{on } A_n \setminus A_{n-1}, \quad n \in \mathbb{N}.$$

Since $(A_n \setminus A_{n-1})_{n \in \mathbb{N}}$ partitions $\tilde{\Omega}$, (u, v) is well-defined. By Lemma 4.72, (u, v) solves (4.5) on $[0, T]$ pointwise on $\tilde{\Omega}$. $\square$

It remains to construct the driving Wiener process for (u, v) and to establish the quantitative bound of Theorem 4.6, completing its proof.

Since $\bar{\mathcal{W}}_{n+1} = \bar{\mathcal{W}}_n$ on $[0, \bar{\tau}_n]$ $\mathbb{P}$ -a.s. (by construction, Section 4.2.4, the same compatibility used for $(\bar{u}_n, \bar{v}_n)$ above), the sequence $(\bar{\mathcal{W}}_n)_{n \in \mathbb{N}}$ satisfies the gluing compatibility (4.27) of Lemma 4.71 with the same stopping times $\bar{\tau}_n$. Gluing exactly as in the preceding proof yields a process $\mathcal{W}$ on $\tilde{\Omega}$ with $\mathbb{P}$ -a.s. continuous paths, progressively measurable, and $\mathcal{W} = \bar{\mathcal{W}}_n$ on $[0, T]$ whenever $\omega \in A_n$. Since each $\bar{\mathcal{W}}_n$ is a cylindrical Wiener process on $\mathcal{H}$ over $\bar{\mathfrak{A}}_n$ and $\mathcal{W}$ agrees with $\bar{\mathcal{W}}_n$ on $[0, T] \times A_n$ for every n with $\mathbb{P}(\bigcup_n A_n) = \mathbb{P}(\tilde{\Omega}) = 1$, $\mathcal{W}$ is a cylindrical Wiener process on $\mathcal{H}$ over the restriction of $\mathfrak{A}$ to $\tilde{\Omega}$. Together with (u, v) , this gives the martingale solution $(\mathfrak{A}, \mathcal{W}, (u, v))$ asserted in Theorem 4.6.

For the quantitative bound, fix $\omega \in \tilde{\Omega}$ and n with $\omega \in A_n$. By the same gluing compatibility, $(\bar{u}_m, \bar{v}_m)|_{[0, T]} = (\bar{u}_n, \bar{v}_n)|_{[0, T]} = (u, v)$ for every $m \geq n$ (each level only extends the solution on $[0, \bar{\tau}_{m-1}] \supseteq [0, T]$, never alters it), so the sequences $\sup_{t \in [0, T]} \|\bar{u}_m(t)\|_{L \log L_{\mathcal{O}}}$, $\sup_{t \in [0, T]} \|\nabla \bar{v}_m(t)\|_{L^2_{\mathcal{O}}}^2$, and $\int_0^T \|\nabla \bar{v}_m(t)\|_{H^1_{\mathcal{O}}}^2 dt$ are eventually constant, hence convergent, at every $\omega \in \tilde{\Omega}$, to the corresponding quantities for (u, v) . Since $\mathbb{P}(\tilde{\Omega}) = 1$, Fatou's lemma applied to these three

nonnegative sequences, together with the bound of Proposition 4.70 (uniform in m), gives

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t)\|_{L \log L_{\mathcal{O}}} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v(t)\|_{L^2}^2 \right] + \mathbb{E} \left[\int_0^T \|\nabla v(t)\|_{H^1}^2 dt \right] \leq C,$$

which is the claimed bound of Theorem 4.6.

4.3 Discussion of the Result and Outlook

In comparison to [HMT22a] we needed stronger assumptions on the initial functions (u_0, v_0) .³ We obtained the same type of a priori energy bound as [HMT22a], and, as in that reference, our result is stated in dimension $d = 1$.

The natural next steps are pathwise uniqueness, following the strategy of [HMT22b] for the production model, and the extension to higher dimensions $d \in \{2, 3\}$. The latter would require replacing the Sobolev embedding $H^1(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$, used throughout this thesis and valid only for $d = 1$, by $W^{1,r}(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$ for $r > d$ (Lemma 2.13(iii)), which in turn would require revisiting every estimate in Section 4.2 that currently relies on the embedding specific for $d = 1$.

³Specifically, Assumption 4.5 requires $u_0 \in H^\alpha(\mathcal{O})$ with $\mathbb{E}[\|u_0\|_{H^\alpha}^2] < \infty$, rather than merely $u_0 \in L^2(\mathcal{O})$: this stronger requirement is used to construct the compact set $L_\varepsilon \subset L^2(\mathcal{O})$ in the proof of Lemma 4.26 via the compact embedding $H^\alpha(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, which is how tightness of the initial values $\mathfrak{T}_n \xi(0)$ is obtained there.

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Declaration of Authorship

I hereby declare that this thesis titled *A stochastic PDE system for chemotaxis* and the work presented in it are my own. I confirm that this work was carried out independently and only with the resources indicated. All passages taken from external sources, whether verbatim or paraphrased, are clearly marked as such. This thesis has not previously been submitted, in the same or similar form, to any other examiner as part of an examination, nor has it been published.

Braunlage, August 18, 2026

Laurin Rodacker